Decomposition of solutions and the Shapley value[☆]

André Casajus[a,*], Frank Huettner[b]

[a] *HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany*
[b] *ESMT European School of Management and Technology, Schlossplatz 1, 10178 Berlin, Germany*

Abstract

We suggest foundations for the Shapley value and for the naïve solution, which assigns to any player the difference between the worth of the grand coalition and its worth after this player left the game. To this end, we introduce the decomposition of solutions for cooperative games with transferable utility. A decomposer of a solution is another solution that splits the former into a direct part and an indirect part. While the direct part (the decomposer) measures a player's contribution in a game as such, the indirect part indicates how she affects the other players' direct contributions by leaving the game. The Shapley value turns out to be unique decomposable decomposer of the naïve solution.

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1. Introduction

The Shapley value (Shapley, 1953) is probably the most eminent one-point solution concept for cooperative games with transferable utility (TU games). Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985). Hart and Mas-Colell (1989) suggest an indirect characterization as marginal contributions of a potential (function).[1] Roth (1977) shows that the Shapley value can be understood as a von Neumann-Morgenstern utility. As a contribution to the Nash program, which aims at building bridges between cooperative and non-cooperative game theory, Pérez-Castrillo and Wettstein (2001) implement the Shapley value as the outcomes

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[*] Corresponding author.
Email addresses: mail@casajus.de (André Casajus), mail@frankhuettner.de (Frank Huettner)
URL: www.casajus.de (André Casajus), www.frankhuettner.de (Frank Huettner)

[1] Calvo and Santos (1997) and Ortmann (1998) generalize the notion of a potential.

of the sub-game perfect equilibria of a combined bidding and proposing mechanism, which is modeled by a non-cooperative extensive form game.[2]

Among the solution concepts for TU games, the Shapley value can be viewed as *the* measure of the players' *own* productivity in a game. This view is strongly supported by Young's (1985) characterization by three properties: efficiency, strong monotonicity, and symmetry. Efficiency says that the worth generated by the grand coalition is distributed among the players. Strong monotonicity requires a player's payoff to increase weakly whenever *her* productivity, measured by *her* marginal contributions to *all coalitions* of the other players, weakly increases. Symmetry ensures that equally productive players obtain the same payoff.

A perhaps naïve way to measure a particular player's productivity within a game is to only look at the marginal contribution of this player to the coalition of *all others*, which we address as the "naïve solution". This solution, however, is problematic for (at least) two reasons. First, in general, the naïve payoffs do not sum up to the worth generated by the grand coalition. Hart and Mas-Colell (1989) use this fact as a motivation of the potential approach to the Shapley value. Second and not less important, one could argue that every player's presence is necessary for generating the naïve payoff of any given player, and therefore the productivity gains reflected in that payoff should be partly attributed to the others.

In order to tackle the second problem of the naïve solution mentioned above, we suggest the decomposition of solutions. A solution ψ decomposes a solution φ if it splits φ into direct and indirect contributions in the following sense. A particular player's payoff for φ is the sum of her payoff for ψ (direct contribution) and what the other players gain or lose under ψ when this particular player leaves the game (indirect contribution). That is, the indirect contributions reflect what a player contributes to the other players' direct contributions. We say that a solution is decomposable if there exists a decomposer, i.e., a solution that decomposes it.

We show that the Shapley value is the unique decomposable decomposer of the naïve solution (Theorem 3). The naïve solution thus conveys interesting information about the Shapley value; and its decomposition may be viewed as a rationale for the naïve solution in terms of the Shapley value. Vice versa, the Shapley value emerges as the natural decomposition of a player's marginal contribution to all other players in the sense that the Shapley value itself can be further rationalized in terms of some underlying solution.

We answer the question of which solutions are decomposable by showing that decomposability is equivalent to a number of other well-known properties of solutions: balanced contributions (Myerson, 1980), path independence (Hart and Mas-Colell, 1989), consistency with the Shapley value (Calvo and Santos, 1997), and admittance of a potential (Calvo and Santos, 1997; Ortmann, 1998) (Theorem 4).

We further establish that amongst all the decomposers of a decomposable solution, there is one and only one that is itself decomposable. It follows immediately that, starting with any decomposable solution φ, there is a unique sequence in which each term is a decomposer of

[2] Ju and Wettstein (2009) suggest a class of bidding mechanisms that implement several solution concepts for TU games.

its predecessor (Theorem 7). We call this sequence the "resolution" of φ. Resolutions allow us to capture higher-order contributions, where for instance the third-order contribution captures what player i contributes to player j's contribution to player k's payoff. We explore the structure of higher-order decomposers using higher-order contributions (Theorem 12).

The remainder of this paper is organized as follows. In Section 2, we give basic definitions and notation. In the third section, we formalize and study the notions of decomposition and decomposability outlined above and present our new rationale for the naïve solution and the Shapley value. The fourth section investigates the notion of decomposability. The fifth section relates higher-order decompositions to higher-order contributions. Some remarks conclude the paper. All proofs are contained in the appendices.

2. Basic definitions and notation

A (TU) game on a finite player set N is given by a **characteristic function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. The set of all games on N is denoted by $\mathbb{V}(N)$. Let $\mathcal{N}$ denote the set of all finite player sets.[3] The cardinalities of $S, T, N, M \in \mathcal{N}$ are denoted by s, t, n, and m, respectively.

For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}(N)$ given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a **unanimity game**. As pointed out in Shapley (1953), these unanimity games form a basis of the vector space[4] $\mathbb{V}(N)$, i.e., any $v \in \mathbb{V}(N)$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T. \tag{1}$$

where the **Harsanyi dividends** $\lambda_T(v)$ can be determined recursively via $\lambda_T(v) = v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v)$ for all $T \subseteq N$, $T \neq \emptyset$ (see Harsanyi, 1959).

A **solution/value** is an operator φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathbb{V}(N)$, $N \in \mathcal{N}$. The **Shapley value** (Shapley, 1953) distributes the dividends $\lambda_T(v)$ equally among the players in T, i.e.,

$$\mathrm{Sh}_i(v) := \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t} \tag{2}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. A solution is **efficient** if $\sum_{i \in N} \varphi_i(v) = v(N)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$.

In this paper, we consider situations where some players leave the game. For $v \in \mathbb{V}(N)$ and $M \subseteq N$, the **restriction** of v to M is denoted by $v|_M \in \mathbb{V}(M)$ and is given by $v|_M = v(S)$ for all $S \subseteq M$; for $v \in \mathbb{V}(N)$ and $M \subseteq N$, the game without the players in M, $v^{-M} \in \mathbb{V}(N \setminus M)$, is given by $v^{-M} = v|_{N \setminus M}$. Instead of $v^{-\{i\}}$, we write v^{-i}.

[3] We assume that the player sets are subsets of some given countably infinite set $\mathfrak{U}$, the universe of players; $\mathcal{N}$ denotes the set of all finite subsets of $\mathfrak{U}$.

[4] For $v, w \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, the games $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$.

3. Decomposing the naïve solution yields the Shapley value

The **naïve solution**, Nï, is given by

$$\text{Nï}_i\left(v\right) := v\left(N\right) - v^{-i}\left(N \setminus \{i\}\right) \tag{3}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}\left(N\right)$, and $i \in N$. It reflects a player i's productivity in a naïve fashion by asking what happens to the worth generated by the grand coalition when i leaves the game.

This naïve solution, however, is problematic for (at least) two reasons. First, in general, the naïve payoffs do not sum up to the worth generated by the grand coalition. Second and possibly more important, the difference $v\left(N\right) - v^{-i}\left(N \setminus \{i\}\right)$ cannot only be attributed to i but also requires the cooperation of players from $N \setminus \{i\}$. Consequently, we are interested in a solution ψ that distributes $v\left(N\right) - v^{-i}\left(N \setminus \{i\}\right)$ among all players,

$$v\left(N\right) - v^{-i}\left(N \setminus \{i\}\right) = \sum_{\ell \in N} \left[\psi_\ell\left(v\right) - \psi_\ell\left(v^{-i}\right)\right]$$

where $\psi_i\left(v^{-i}\right) = 0$. In other words, we are interested in a solution that decomposes the naïve solution in the above sense, i.e.,

$$\text{Nï}_i\left(v\right) = \psi_i\left(v\right) + \sum_{\ell \in N\setminus\{i\}} \left[\psi_\ell\left(v\right) - \psi_\ell\left(v^{-i}\right)\right]$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}\left(N\right)$, and $i \in N$. This motivates the following definition.

Definition 1. *A solution ψ is a* ***decomposer*** *of the solution φ if*

$$\varphi_i\left(v\right) = \psi_i\left(v\right) + \sum_{\ell \in N\setminus\{i\}} \left[\psi_\ell\left(v\right) - \psi_\ell\left(v^{-i}\right)\right] \tag{4}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}\left(N\right)$, and $i \in N$. A solution φ is called ***decomposable*** *if there exists a decomposer ψ of φ.*

A decomposer ψ of φ breaks up player i's payoff $\varphi_i\left(v\right)$ into a direct part $\psi_i\left(v\right)$ and an indirect part $\sum_{\ell \in N\setminus\{i\}} \left[\psi_\ell\left(v\right) - \psi_\ell\left(v^{-i}\right)\right]$. The indirect part indicates how much player i contributes to the other players' direct parts. Thus, if a decomposer ψ of φ exists, then it provides a kind of rationale or foundation for φ.

However, a solution φ that is decomposable has many different decomposers, so that there is ambiguity about which of them to select. It turns out that there is a unique one amongst all the decomposers of φ that is itself decomposable, i.e., that is rationalizable in terms of some underlying solution. Precisely, we have:

Proposition 2. *If a solution φ is decomposable, then it has one, and only one, decomposer ψ that is itself decomposable.*

In view of Proposition 2, we may talk of *the* (decomposable) decomposer $\psi^{(1)}$ of any decomposable solution φ; and indeed *the* decomposer $\psi^{(2)}$ of $\psi^{(1)}$, $\psi^{(3)}$ of $\psi^{(2)}$, and so on ad infinitum. We shall examine this full sequence later when we discuss the "resolution" of φ. But we need look at just the first term in the sequence to rationalize the naïve solution in terms of the Shapley value.

Theorem 3. *The Shapley value is the unique decomposable decomposer of the naïve solution.*

It is evident that any efficient solution is a decomposer of the naïve solution.[5] The main point of Theorem 3 is that the Shapley value is distinguished amongst them as the only one that is decomposable.

At first glance, Theorem 3 seems to imply that the Shapley value is determined by the "last" marginal contributions. A deeper look, however, reveals that this, of course, is not the case. By considering the effects of players' leaving the game on other players' payoffs, we relate the payoff for a game to payoffs in games with smaller player sets. Since solutions apply to games with any number of players, in the end, all marginal contributions are taken into account.

4. Decomposability

In the previous section, we have highlighted the Shapley value as a the unique decomposable decomposer of the naïve solution. This triggers the question of which solutions are decomposable? To answer this question, we establish that the notion of decomposability is equivalent to a number of well-known properties.

Theorem 4. *The following properties of a solution φ are equivalent:*

(i) Decomposability: The solution φ is decomposable.

(ii) Balanced contributions (Myerson, 1980): For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j \in N$, we have

$$\varphi_i(v) - \varphi_i\left(v^{-j}\right) = \varphi_j(v) - \varphi_j\left(v^{-i}\right).$$

(iii) Path independence (Hart and Mas-Colell, 1989): For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and all bijections $\rho, \rho' : N \to \{1, \ldots, n\}$, we have

$$\sum_{i \in N} \varphi_i\left(v^{-S_i(\rho)}\right) = \sum_{i \in N} \varphi_i(v^{-S_i(\rho')}),$$

where $S_i(\rho) := \{\ell \in N \mid \rho(\ell) > \rho(i)\}$.

[5] Indeed, let ψ be any efficient solution. Then, $v(N) - v^{-i}(N \setminus \{i\}) = \sum_{\ell \in N} \psi_\ell(v) - \sum_{\ell \in N \setminus \{i\}} \psi_\ell\left(v^{-i}\right) = \psi_i(v) + \sum_{\ell \in N \setminus \{i\}} \left[\psi_\ell(v) - \psi_\ell\left(v^{-i}\right)\right]$.

(iv) *Admittance of a potential (Calvo and Santos, 1997; Ortmann, 1998): There exits a mapping* $P : \cup_{N \in \mathcal{N}} \mathbb{V}(N) \to \mathbb{R}$ *such that*

$$\varphi_i(v) = P(v) - P\left(v^{-i}\right)$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, *and* $i \in N$.

(v) *Consistency with the Shapley value (Sánchez S., 1997; Calvo and Santos, 1997): For all* $N \in \mathcal{N}$ *and* $v \in \mathbb{V}(N)$, *we have* $\varphi(v) = \mathrm{Sh}(v^{\varphi})$, *where* $v^{\varphi} \in \mathbb{V}(N)$ *is given by*

$$v^{\varphi}(S) = \sum_{\ell \in S} \varphi_{\ell}(v|_S) \qquad \text{for all } S \subseteq N. \tag{5}$$

Balanced contributions states that player j suffers/gains from the removal of player i by the same amount as the other way around. Path independence can be interpreted as follows. Sequentially buying players out of the game by paying them their payoffs according to φ, costs the same independent of the order in which the players are bought out. A solution admits a potential if it is the differential of some potential function on the domain of all games. A solution φ is consistent according to the definition in Theorem 4(v) if it can be obtained as the Shapley value of an auxiliary game. In the auxiliary game, the worth generated by a coalition equals the sum of its players' payoffs according to φ in the game restricted to this coalition.

Important examples of solutions that satisfy the properties of Theorem 4 are the family of semivalues (Dubey et al., 1981; Calvo and Santos, 1997). Besides the Shapley value, this class contains the Banzhaf-Owen value (Banzhaf, 1965; Owen, 1975; Dubey and Shapley, 1979). Solutions that are efficient but differ from the Shapley value as for example the egalitarian Shapley values (Joosten, 1996) or the equal surplus division value (Driessen and Funaki, 1991) do not satisfy the properties of Theorem 4.

To prove Theorem 4, we shall show the equivalence of (i) and (iv), since the rest follows from Calvo and Santos (1997, Corollary 3.4). This equivalence is, in turn, driven by the fact that a solution ψ is a decomposer of the solution φ if and only if

$$P^{\varphi}(v) = \sum_{i \in N} \psi_i(v)$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, where P^{φ} denotes the **zero-normalized potential** of φ.[6] This further implies that a decomposer ψ of a solution φ is not unique. In fact, any solution ψ' such that $\sum_{\ell \in N} \psi'_{\ell}(v) = \sum_{\ell \in N} \psi_{\ell}(v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ also decomposes φ. As stated in Proposition 2, however, there can be only one decomposer of φ that is decomposable itself. This is further elaborated on in the next section.

[6] A potential P is zero-normalized if $P(\emptyset) = 0$ for $\emptyset \in \mathbb{V}(\emptyset)$.

5. Higher-order decompositions

In this section, we study decompositions of decomposers. From Proposition 2, we already know that there exists at most one decomposable decomposer of a decomposable solution. The next proposition clarifies the existence and the structure of such a decomposer.

Proposition 5. *If a solution φ is decomposable, then its unique decomposable decomposer ψ is given by*

$$\psi_i(v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v^\varphi)}{t^2} \tag{6}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, where v^φ is defined in (5).

As an immediate consequence of this proposition, a solution is decomposable if and only if it has a decomposable decomposer. This gives rise to study the sequence of decomposable decomposers and to the following definition.

Definition 6. *A* ***resolution*** *of a solution φ is a sequence $\left(\varphi^{(k)}\right)_{k \in \mathbb{N}}$ of solutions such that $\varphi^{(0)} = \varphi$ and $\varphi^{(k+1)}$ is a decomposer $\varphi^{(k)}$ for all $k \in \mathbb{N}$. If a resolution exists for a solution, then the latter is called* ***resolvable****.*

As an immediate consequence of Proposition 5, resolvability is equivalent to decomposability. Moreover, a resolution, if it exists, is unique. Therefore, the solution $\varphi^{(k)}$ is called the **kth decomposer of** φ. The next result specifies that the formulae for the higher-order decomposers are similar to (6).

Theorem 7. *Let the solution φ be decomposable.*

(i) There exists a unique resolution $\left(\varphi^{(k)}\right)_{k \in \mathbb{N}}$ of φ.

(ii) The resolution of φ is given by

$$\varphi_i^{(k)}(v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v^\varphi)}{t^{k+1}} \tag{7}$$

for all $k \in \mathbb{N}$, $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$.

(iii) The resolution of the Shapley value is given by

$$\mathrm{Sh}_i^{(k)}(v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t^{k+1}}$$

for all $k \in \mathbb{N}$, $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$.

(iv) The resolution of φ is given by

$$\varphi^{(k)}(v) = \mathrm{Sh}^{(k)}(v^\varphi)$$

for all $k \in \mathbb{N}$, $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$.

(v) The resolution of φ is given by

$$\varphi^{(k)}(v) = \mathrm{Sh}^{(k-1)}\left(v^{P^\varphi}\right)$$

for all $k \in \mathbb{N}$, $k > 0$, $N \in \mathcal{N}$, and $v \in \mathbb{V}(N)$, where $v^{P^\varphi} \in \mathbb{V}(N)$ is given by $v^{P^\varphi}(S) = P^\varphi(v|_S)$ for all $S \subseteq N$.

Note that $\varphi^{(0)} = \varphi$, which follows from the consistency with the Shapley value of φ (Theorem 4) and (2). The theorem has two implications. First, for given $k > 0$, the per capita dividends of greater coalitions have a lower influence on the payoff than those of small coalitions. This reflects that dividends of greater coalitions are more "endangered" by players leaving the game. Second, with increasing order of decomposition, the per capita dividends of non-singleton coalitions have a lower influence on the payoffs. As an explanation we suggest that higher-order decomposers sum up to the original value taking into account increasingly more indirect effects. The extreme case is given in the next corollary.

Corollary 8. *For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $i \in N$, and any decomposable solution φ, we have*

$$\lim_{k\to\infty} \varphi_i^{(k)}(v) = \varphi_i\left(v|_{\{i\}}\right).$$

Next, we will investigate the relationship between higher-order contributions and higher-order decompositions. It is evident that, in accordance with (4), one can start with φ written out in terms of $\varphi^{(1)}$, and successively substitute $\varphi^{(k+1)}$ for $\varphi^{(k)}$ for $k \in \{1, \ldots, \alpha\}$ to arrive at a formula for φ in terms of $\varphi^{(\alpha)}$. It is further evident that the formula will entail several higher-order contributions of individuals, i.e., for sequences of individuals $i_1, i_2, \ldots, i_\ell$ one will need to consider the contribution of i_ℓ to $i_{\ell-1}$'s contribution to ... to i_2's contribution to i_1. This is made precise in Theorem 12.

For example, if χ is the decomposer of ψ and ψ is the decomposer of φ, then substitution of $\psi_i(v)$ by $\chi_i(v) + \sum_{\ell \in N\setminus\{i\}} [\chi_\ell(v) - \chi_\ell(v^{-i})]$ in (4) yields

$$\begin{aligned}\varphi_i(v) = \chi_i(v) &+ 3 \cdot \sum_{\ell \in N\setminus\{i\}} \left[\chi_\ell(v) - \chi_\ell\left(v^{-i}\right)\right] \\ &+ \sum_{j \in N\setminus\{i\}} \sum_{k \in N\setminus\{i,j\}} \left[\chi_k(v) - \chi_k\left(v^{-j}\right)\right] - \left[\chi_k\left(v^{-i}\right) - \chi_k\left(v^{-\{j,i\}}\right)\right],\end{aligned}$$

where player i makes a contribution to player j's contribution to player k's payoff according to χ,

$$\left[\chi_k(v) - \chi_k\left(v^{-j}\right)\right] - \left[\chi_k\left(v^{-i}\right) - \chi_k\left(v^{-\{j,i\}}\right)\right]. \tag{8}$$

We define higher-order contributions as follows. Let the players i, j, and k be distinct. The contribution of j to k's payoff under the solution φ in the games v and v^{-i} is given by

$$D_{(k,j)}\varphi(v) = \varphi_k(v) - \varphi_k\left(v^{-j}\right)$$

and

$$D_{(k,j)}\varphi\left(v^{-i}\right)=\left[\varphi_k\left(v^{-i}\right)-\varphi_k\left(v^{-\{j,i\}}\right)\right].$$

Further, the contribution of i to j's contribution to k is given by

$$\begin{aligned}D_{(k,j,i)}\varphi(v)&=\left[\varphi_k(v)-\varphi_k\left(v^{-j}\right)\right]-\left[\varphi_k\left(v^{-i}\right)-\varphi_k\left(v^{-\{j,i\}}\right)\right]\\&=D_{(k,j)}\varphi(v)-D_{(k,j)}\varphi\left(v^{-i}\right)\end{aligned}$$

In general, we have the following recursive definition.

Definition 9. *For any solution φ, all $N\subseteq M\in\mathcal{N}$, and $v\in\mathbb{V}(N)$, we set*

$$\varphi_i(v)=0\qquad\text{for all } i\in M\setminus N. \tag{9}$$

Moreover, for all $\alpha\in\mathbb{N}$, $N\subseteq M\in\mathcal{N}$, $v\in\mathbb{V}(N)$, $i\in M$, and $\mathbf{i}=(\mathbf{i}_1,\dots,\mathbf{i}_\alpha)\in M^\alpha$, we define $D_{\mathbf{i}}\varphi(v)$ recursively by

$$\begin{aligned}D_{()}\varphi(v)&:=0,\\D_{(i)}\varphi(v)&:=\varphi_i(v),\\D_{\mathbf{i}}\varphi(v)&:=D_{(\mathbf{i}_1,\dots,\mathbf{i}_{\alpha-1})}\varphi(v)-D_{(\mathbf{i}_1,\dots,\mathbf{i}_{\alpha-1})}\varphi\left(v^{-\mathbf{i}_\alpha}\right).\end{aligned}\tag{10}$$

The following theorem expresses a solution φ as the sum of higher-order contributions $D_{(\mathbf{i},i)}$ of a fixed higher-order decomposer $\varphi^{(\alpha)}$.

Proposition 10. *For any decomposable solution φ, we have*

$$\varphi_i(v)=\sum_{\mathbf{i}\in N^\alpha}D_{(\mathbf{i},i)}\varphi^{(\alpha)}(v)\tag{11}$$

for all $\alpha\in\mathbb{N}$, $N\in\mathcal{N}$, $v\in\mathbb{V}(N)$, and $i\in N$.

To illustrate this result, let $N=\{1,2,3\}$ and $\alpha=2$. Then,

$$\begin{aligned}\varphi_1(v)=&\,D_{(1,1,1)}\varphi^{(2)}(v)+D_{(1,2,1)}\varphi^{(2)}(v)+D_{(1,3,1)}\varphi^{(2)}(v)\\&+D_{(2,1,1)}\varphi^{(2)}(v)+D_{(2,2,1)}\varphi^{(2)}(v)+D_{(2,3,1)}\varphi^{(2)}(v)\\&+D_{(3,1,1)}\varphi^{(2)}(v)+D_{(3,2,1)}\varphi^{(2)}(v)+D_{(3,3,1)}\varphi^{(2)}(v).\end{aligned}$$

At first glance, this formula involves only third-order contributions. In view of Definition 10, however, some of the expressions actually are first-order contributions and second-order contributions. Indeed, we have

$$\begin{aligned}\varphi_1(v)=&\,D_{(1)}\varphi^{(2)}(v)\\&+3\cdot D_{(2,1)}\varphi^{(2)}(v)+3\cdot D_{(3,1)}\varphi^{(2)}(v)+D_{(2,3,1)}\varphi^{(2)}(v)+D_{(3,2,1)}\varphi^{(2)}(v).\end{aligned}$$

In this formula, all (higher-order) contributions have a natural interpretation. In order to capture this observation more generally, we invoke the following definition.

Definition 11. *For all $\alpha \in \mathbb{N}$ and $N \in \mathcal{N}$ define the set of all sequences with members from N of length α with distinct members by*

$$N^{[\alpha]} := \{\mathbf{i} \in N^{\alpha} \mid \mathbf{i}_k \neq \mathbf{i}_\ell \text{ for all } k, \ell \in \{1, \ldots, \alpha\}, \; k \neq \ell\},$$

where N^{α} denotes the set of all sequences with members from N of length α. Note that $N^{[0]} = N^0 = \{()\}$ for all $N \in \mathcal{N}$.

The following theorem expresses a solution φ as the sum of genuine (higher-order) contributions $D_{(\mathbf{i},i)}$ of a fixed higher-order decomposer $\varphi^{(\alpha)}$.

Theorem 12. *Fix $\alpha \in \mathbb{N}$. For any decomposable solution φ, we have*

$$\varphi_i(v) = \sum_{t=0}^{\alpha} g(t, \alpha) \cdot \sum_{\mathbf{i} \in (N \setminus \{i\})^{[t]}} D_{(\mathbf{i},i)} \varphi^{(\alpha)}(v) \tag{12}$$

for $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, where $g(t, \alpha) > 0$ and

$$g(t, \alpha) = \frac{1}{t!} \cdot \sum_{s=0}^{t} (-1)^{t-s} \cdot \binom{t}{s} \cdot (s+1)^{\alpha}. \tag{13}$$

Note that the summation index in (12) is in fact bounded above by $\min\{\alpha, n-1\}$. For $t > n-1$, the right-most sum in (12) is empty because $(N \setminus \{i\})^{[t]} = \emptyset$.

6. Concluding remarks

We introduce the decomposition of a solution by another solution (the decomposer) into a direct part and an indirect part. The indirect part indicates how much a player contributes to another player's payoff according to the decomposer.

We characterize the solutions that can be decomposed and use this insight in order to provide a new foundation of the naïve solution and interesting insights regarding the Shapley value. In particular, the Shapley value is the unique decomposable decomposer of the naïve solution.

Taking the idea of decomposition one step further leads to asking how much one player contributes to how much a second player contributes to a third player's payoff in a game. This leads to the notion of higher-order contributions and higher-order decompositions of solutions.

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Appendix A. Proof of Theorem 4

It suffices to show that (i) is equivalent to (iv), since it has been proved in Calvo and Santos (1997, Corollary 3.4) that (ii), (iii), (iv), and (v) are equivalent.

Let $P : \mathbb{V} \to \mathbb{R}$ be a potential for the solution φ. Let the solution ψ^P be given by $\psi_i^P(v) = |N|^{-1} \cdot P(v)$ for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. By construction, we have

$$\begin{aligned}
\psi_i^P(v) + \sum_{\ell \in N \setminus \{i\}} \left(\psi_\ell^P(v) - \psi_\ell^P\left(v^{-i}\right)\right) &= \sum_{\ell \in N} \psi_\ell^P(v) - \sum_{\ell \in N \setminus \{i\}} \psi_\ell^P\left(v^{-i}\right) \\
&= P(v) - P\left(v^{-i}\right) \\
&= \varphi_i(v)
\end{aligned}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. That is, ψ^P decomposes φ.

Let ψ be a decomposer of the solution φ. Let the mapping $P^\psi : \mathbb{V} \to \mathbb{R}$ be given by $P^\psi(v) = \sum_{\ell \in N} \psi_\ell(v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$. By construction, we have

$$\begin{aligned}
P^\psi(v) - P^\psi\left(v|_{N \setminus \{i\}}\right) &= \sum_{\ell \in N} \psi_\ell(v) - \sum_{\ell \in N \setminus \{i\}} \psi_\ell\left(v^{-i}\right) \\
&= \psi_i(v) + \sum_{\ell \in N \setminus \{i\}} \left(\psi_\ell(v) - \psi_\ell\left(v^{-i}\right)\right) \\
&= \varphi_i(v)
\end{aligned}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. That is, P^ψ is a potential for φ.

Appendix B. Lemma and proof

The following lemma will be useful for the proofs of Proposition 5 and Theorem 7.

Lemma 13. *For all $k \in \mathbb{N}$, $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, we have*

$$\sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t^{k+1}} + \sum_{\ell \in N \setminus \{i\}} \left[\sum_{T \subseteq N : \ell \in T} \frac{\lambda_T(v)}{t^{k+1}} - \sum_{T \subseteq N \setminus \{i\} : \ell \in T} \frac{\lambda_T\left(v^{-i}\right)}{t^{k+1}}\right] = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t^k}.$$

This can be seen as follows: For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$,

$$\begin{aligned}
\sum_{\ell \in N} \sum_{T \subseteq N : \ell \in T} \frac{\lambda_T(v)}{t^{k+1}} - \sum_{\ell \in N \setminus \{i\}} \sum_{T \subseteq N \setminus \{i\} : \ell \in T} \frac{\lambda_T\left(v^{-i}\right)}{t^{k+1}} &= \sum_{T \subseteq N} \sum_{\ell \in T} \frac{\lambda_T(v)}{t^{k+1}} - \sum_{T \subseteq N \setminus \{i\}} \sum_{\ell \in T} \frac{\lambda_T\left(v^{-i}\right)}{t^{k+1}} \\
&= \sum_{T \subseteq N} \frac{\lambda_T(v)}{t^k} - \sum_{T \subseteq N \setminus \{i\}} \frac{\lambda_T(v)}{t^k},
\end{aligned}$$

where the last equation uses the obvious fact that $\lambda_T(v) = \lambda_T\left(v^{-i}\right)$ for $T \subseteq N \setminus \{i\}$.

Appendix C. Proof of Proposition 5

Uniqueness: Let the solution φ be decomposable and let ψ and ψ' be decomposable decomposers of φ. We show $\psi(v) = \psi'(v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ by induction on n.

Induction basis: For $n = 1$, the claim is immediate from (4).

Induction hypothesis : Suppose $\psi(v) = \psi'(v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ such that $n \leq t$.

Induction step: Let $N \in \mathcal{N}$ be such that $|N| = t+1$. For $v \in \mathbb{V}(N)$ and $i \in N$, we have

$$\psi_i(v) + \sum_{\ell \in N \setminus \{i\}} \left(\psi_\ell(v) - \psi_\ell\left(v^{-i}\right)\right) \overset{(4)}{=} \varphi_i(v) \overset{(4)}{=} \psi'_i(v) + \sum_{\ell \in N \setminus \{i\}} \left(\psi'_\ell(v) - \psi'_\ell\left(v^{-i}\right)\right).$$

Hence, the induction hypothesis implies

$$\sum_{\ell \in N} \psi_\ell(v) = \sum_{\ell \in N} \psi'_\ell(v). \tag{C.1}$$

Since ψ and ψ' are decomposable and by Theorem 4, both ψ and ψ' satisfy the balanced contributions property. Hence, we have

$$\psi_i(v) - \psi_i\left(v^{-j}\right) = \psi_j(v) - \psi_j\left(v^{-i}\right) \qquad \text{for all } j \in N \setminus \{i\}. \tag{C.2}$$

Summing up (C.2) over all $j \in N \setminus \{i\}$ gives

$$(n-1) \cdot \psi_i(v) - \sum_{j \in N \setminus \{i\}} \psi_i\left(v^{-j}\right) = \sum_{j \in N \setminus \{i\}} \left(\psi_j(v) - \psi_j\left(v^{-i}\right)\right)$$

and therefore

$$n \cdot \psi_i(v) = \sum_{\ell \in N} \psi_\ell(v) - \sum_{j \in N \setminus \{i\}} \psi_j\left(v^{-i}\right) + \sum_{j \in N \setminus \{i\}} \psi_i\left(v^{-j}\right). \tag{C.3}$$

Analogously, we obtain

$$n \cdot \psi'_i(v) = \sum_{\ell \in N} \psi'_\ell(v) - \sum_{j \in N \setminus \{i\}} \psi'_j\left(v^{-i}\right) + \sum_{j \in N \setminus \{i\}} \psi'_i\left(v^{-j}\right). \tag{C.4}$$

Finally, (C.1), (C.3), (C.4), and the induction hypothesis entail $\psi_i(v) = \psi'_i(v)$.

Existence: Let the solution φ be decomposable and let the solution ψ be given by (6). For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, we obtain

$$\psi_i(v) + \sum_{\ell \in N \setminus \{i\}} \left(\psi_\ell(v) - \psi_\ell\left(v^{-i}\right)\right) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v^\varphi)}{t} = \mathrm{Sh}_i(v^\varphi) = \varphi_i(v),$$

where the first equation follows from Lemma 13 (in conjunction with the obvious fact that $(v^\varphi)^{-i} = (v^{-i})^\varphi$), the second equation comes from (2), and the last equation follows from Theorem 4(v). Thus, ψ is a decomposer of φ.

It remains to establish that ψ is decomposable. By (i) and (iv) of Theorem 4, this will follow if ψ admits a potential. It is easy to check that the mapping $P^{\psi} : \mathbb{V} \to \mathbb{R}$ given by

$$P^{\psi}(v) = \sum_{T \subseteq N : T \neq \emptyset} \frac{\lambda_T(v^{\varphi})}{t^2}$$

for all $N \in \mathcal{N}$, and $v \in \mathbb{V}(N)$ is a potential for ψ.

Appendix D. Proof of Theorem 3

As we saw (Footnote 5), any efficient solution decomposes the naïve solution, and hence the Shapley value Sh is a decomposer of the naïve solution (since Sh is efficient). But by Hart and Mas-Colell (1989, Theorem A), a potential exists for Sh. So, by the equivalence of (iv) and (i) in Theorem 4, Sh is itself decomposable. Then, by Proposition 2, it is the only decomposer of the naïve solution that is decomposable.

Appendix E. Proof of Theorem 7

(i) This is an immediate consequence of Proposition 5.

(ii) We show the claim by induction on k.

Induction basis: For $k = 0$, the claim follows from Theorem 4(v) and (2). For $k = 1$, the claim is established in Proposition 5.

Induction hypothesis (IH): Suppose the resolution of a decomposable φ is given by

$$\varphi_i^{(k)}(v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v^{\varphi})}{t^{k+1}}$$

for all $k \leq K$, $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$.

Induction step: Let the solution ψ be given by

$$\psi_i(v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v^{\varphi})}{t^{K+2}} \tag{E.1}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. We obtain

$$\begin{aligned}
&\psi_i(v) + \sum_{\ell \in N \setminus \{i\}} \left[\psi_\ell(v) - \psi_\ell\left(v^{-i}\right)\right] \\
&= \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v^\varphi)}{t^{K+2}} + \sum_{\ell \in N \setminus \{i\}} \left[\sum_{T \subseteq N : \ell \in T} \frac{\lambda_T(v^\varphi)}{t^{K+2}} - \sum_{T \subseteq N \setminus \{i\} : \ell \in T} \frac{\lambda_T\left(\left(v^{-i}\right)^\varphi\right)}{t^{K+2}}\right] \\
&\overset{(6)}{=} \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v^\varphi)}{t^{K+2}} + \sum_{\ell \in N \setminus \{i\}} \left[\sum_{T \subseteq N : \ell \in T} \frac{\lambda_T(v^\varphi)}{t^{K+2}} - \sum_{T \subseteq N \setminus \{i\} : \ell \in T} \frac{\lambda_T\left(\left(v^\varphi\right)^{-i}\right)}{t^{K+2}}\right] \\
&= \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v^\varphi)}{t^{K+1}} \\
&\overset{IH}{=} \varphi_i^{(K)}(v)
\end{aligned}$$

where the third equation follows from Lemma 13. This establishes that ψ is a decomposer of $\varphi^{(K)}$. Remains to show that ψ is decomposable itself. By Theorem 4, it suffices to show that ψ admits a potential. In view of (E.1), one easily checks that the mapping $P^\psi : \mathbb{V} \to \mathbb{R}$ given by

$$P^\psi(v) = \sum_{T \subseteq N : T \neq \emptyset} \frac{\lambda_T(v^\varphi)}{t^{K+2}}$$

for all $N \in \mathcal{N}$, and $v \in \mathbb{V}(N)$ is a potential for ψ.

(iii) Since $v^{\mathrm{Sh}} = v$ for all $v \in \mathbb{V}(N)$, this follows from *(ii)* and (2).

(iv) This is immediate from *(iii)* and *(ii)*.

(v)

Appendix F. Proof of Proposition 10

Let φ be a decomposable solution. We proceed by induction on $\alpha \in \mathbb{N}$.

Induction basis: The claim is immediate for $\alpha = 0$.

For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, we have

$$\begin{aligned}
\sum_{j \in N} D_{(j,i)}\varphi^{(1)}(v) &= \sum_{j \in N} \left[D_j\varphi^{(1)}(v) - D_j\varphi^{(1)}\left(v^{-i}\right)\right] \\
&= \sum_{j \in N} \left[\varphi_j^{(1)}(v) - \varphi_j^{(1)}\left(v^{-i}\right)\right] \\
&= \varphi_i^{(1)}(v) + \sum_{j \in N \setminus \{i\}} \left[\varphi_j^{(1)}(v) - \varphi_j^{(1)}\left(v^{-i}\right)\right] \\
&= \varphi_i(v),
\end{aligned}$$

where the last equation follows from the fact that $\varphi = \varphi^{(0)}$ is decomposed by $\varphi^{(1)}$.

Induction hypothesis: For all for $\alpha \in \mathbb{N}$, $1 < \alpha \leq \bar{\alpha}$, $2 \leq \bar{\alpha}$, $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, we have

$$\varphi_i(v) = \sum_{\mathbf{i} \in N^\alpha} D_{(\mathbf{i},i)} \varphi^{(\alpha)}(v).$$

Induction step: For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, we obtain

$$\begin{aligned}
&\sum_{\mathbf{i} \in N^{A+1}} D_{(\mathbf{i},i)} \varphi^{(\bar{\alpha}+1)}(v) \overset{(10)}{=} \sum_{\mathbf{i} \in N^{A+1}} \left[D_{\mathbf{i}} \varphi^{(\bar{\alpha}+1)}(v) - D_{\mathbf{i}} \varphi^{(\bar{\alpha}+1)}\left(v^{-i}\right) \right] \\
&= \sum_{j \in N} \varphi_j^{(1)}(v) - \sum_{j \in N \setminus \{i\}} \varphi_j^{(1)}\left(v^{-i}\right) \\
&= \varphi_i(v)
\end{aligned}$$

where the second equation follows from the induction hypothesis, (9), and the fact that $\varphi^{(1)}$ is resolved by $\left(\left(\varphi^{(\tau)} \right)_{\tau \in \mathbb{N} : \tau > 0} \right)$ and the third equation follows from the assumption that $\varphi^{(1)}$ decomposes $\varphi^{(0)} = \varphi$.

Appendix G. Proof of Theorem 12

We prove the claim by a number of lemmas.

Lemma 14. *For any decomposable solution φ, we have*

$$D_{\mathbf{i}} \varphi(v) = D_{\pi \mathbf{i}} \varphi(v)$$

for all $\alpha \in \mathbb{N}$, $N \subseteq M \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $\mathbf{i} \in M^a$, and all bijections $\pi : \{1, \ldots, \alpha\} \to \{1, \ldots, \alpha\}$, where $\pi \mathbf{i} \in M^a$ is given by $(\pi \mathbf{i})_\ell = \mathbf{i}_{\pi(\ell)}$ for all $\ell \in \{1, \ldots, \alpha\}$.

Proof. Let φ be a decomposable solution. We proceed by induction on $\alpha \in \mathbb{N}$.

Induction basis: The claim is immediate for $\alpha \leq 1$, and $\mathbf{i} = (i, i)$, $i \in M$. For $(i, j) \in M^{[2]}$, we obtain

$$D_{(i,j)} \varphi(v) \overset{(10)}{=} \varphi_i(v) - \varphi_i\left(v^{-j}\right) = \varphi_j(v) - \varphi_j\left(v^{-i}\right) \overset{(10)}{=} D_{(j,i)} \varphi(v),$$

where second equation follows from Theorem 4 and the assumption that φ is decomposable. This shows the claim for $\alpha = 2$.

Induction hypothesis (IH): Let the claim hold for all $\alpha \in \mathbb{N}$, $\alpha \leq \bar{\alpha}$, $2 \leq \bar{\alpha}$, $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $\mathbf{i} \in M^a$, and all bijections $\pi : \{1, \ldots, \alpha\} \to \{1, \ldots, \alpha\}$.

Induction step: For $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $\mathbf{i} \in N^{\bar{\alpha}}$, $i \in N$, and all bijections $\pi : \{1, \ldots, \bar{\alpha}\} \to \{1, \ldots, \bar{\alpha}\}$, we obtain

$$D_{(\mathbf{i},i)} \varphi(v) \overset{(10)}{=} D_{\mathbf{i}} \varphi(v) - D_{\mathbf{i}} \varphi\left(v^{-i}\right) \overset{IH}{=} D_{\pi \mathbf{i}} \varphi(v) - D_{\pi \mathbf{i}} \varphi\left(v^{-i}\right) \overset{(10)}{=} D_{(\pi \mathbf{i},i)} \varphi(v).$$

Further, for $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $\mathbf{i} \in M^{\bar{\alpha}-1}$, and $i, j \in M$, equation (10) entails

$$\begin{aligned} D_{(\mathbf{i},i,j)}\varphi(v) &= D_{(\mathbf{i},i)}\varphi(v) - D_{(\mathbf{i},i)}\varphi\left(v^{-j}\right) \\ &= D_{\mathbf{i}}\varphi(v) - D_{\mathbf{i}}\varphi\left(v^{-i}\right) - \left(D_{\mathbf{i}}\varphi\left(v^{-j}\right) - D_{\mathbf{i}}\varphi\left(v^{-\{i,j\}}\right)\right) \\ &= D_{(\mathbf{i},j)}\varphi(v) - D_{(\mathbf{i},j)}\varphi\left(v^{-i}\right) \\ &= D_{(\mathbf{i},j,i)}\varphi(v). \end{aligned}$$

Hence, the claim holds for all bijections $\pi : \{1, \ldots, \bar{\alpha}+1\} \to \{1, \ldots, \bar{\alpha}+1\}$. □

For all $\alpha \in \mathbb{N}$, $N \in \mathcal{N}$, and $\mathbf{i} \in N^{\alpha}$, we set $\text{car}(\mathbf{i}) := \{\mathbf{i}_\ell \mid \ell \in \{1, \ldots, \alpha\}\}$. Hence, we have $N^{[\alpha]} = \{\mathbf{i} \in N^{\alpha} \mid |\text{car}(\mathbf{i})| = \alpha\}$. Recall that $N^{[0]} = N^0 = \{()\}$.

Lemma 15. *For any decomposable solution φ, we have $D_{(\mathbf{i},i)}\varphi(v) = D_{\mathbf{i}}\varphi(v)$ for all $\alpha \in \mathbb{N}$, $N \subseteq M \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $\mathbf{i} \in M^{\alpha}$, and $i \in M$ such that $i \in$ car$(\mathbf{i})$.*

Proof. Let φ be a decomposable solution and $\alpha \in \mathbb{N}$, $N \subseteq M \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $\mathbf{i} \in M^{\alpha}$, and $i \in N$ such that $i \in \text{car}(\mathbf{i})$. By Lemma 14, we are allowed to assume that $\mathbf{i}_\alpha = i$. By (10), we have

$$\begin{aligned} D_{(\mathbf{i},i)}\varphi(v) &= D_{\mathbf{i}}\varphi(v) - D_{\mathbf{i}}\varphi\left(v^{-i}\right) \\ &= D_{\mathbf{i}}\varphi(v) - \left(D_{(\mathbf{i}_1,\ldots,\mathbf{i}_{\alpha-1})}\varphi\left(v^{-i}\right) - D_{(\mathbf{i}_1,\ldots,\mathbf{i}_{\alpha-1})}\varphi\left(v^{-\{i,\mathbf{i}_\alpha\}}\right)\right) \\ &= D_{\mathbf{i}}\varphi(v), \end{aligned}$$

which proves the claim. □

The following lemma is immediate from Lemmas 14 and 15.

Lemma 16. *If φ is decomposable, then $D_{\mathbf{i}}\varphi(v) = D_{\mathbf{j}}\varphi(v)$ for all $\alpha \in \mathbb{N}$, $N \subseteq M \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $\mathbf{i}, \mathbf{j} \in M^{\alpha}$ such that* car$(\mathbf{i}) =$ car$(\mathbf{j})$.

For any decomposable solution φ and all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $S \in \mathcal{N}$, we define *higher-order differences* $D_S\varphi(v)$ by

$$D_S\varphi(v) = D_{\mathbf{i}}\varphi(v) \tag{G.1}$$

with $\mathbf{i} \in M^{\alpha}$, $\alpha \in \mathbb{N}$ such that $\text{car}(\mathbf{i}) = S$. In view of Lemma 16, $D_S\varphi(v)$ is well-defined.

Lemma 17. *For any decomposable solution φ, we have*

$$\varphi_i(v) = \sum_{S \subseteq N\setminus\{i\}: 0 \le s \le \alpha} f(s,\alpha) \cdot D_{S\cup\{i\}}\varphi^{(\alpha)}(v)$$

for all $\alpha \in \mathbb{N}$, $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, where

$$f(s,\alpha) = \sum_{t=0}^{s} \left[(-1)^{s-t} \cdot \binom{s}{t} \cdot (t+1)^{\alpha}\right]. \tag{G.2}$$

Proof. The claim follows from Lemmas 10 and 16, and (G.1) as follows. For $\alpha \in \mathbb{N}$, $N \in \mathcal{N}$, $S \subseteq N$, $s \leq \alpha$, and $i \in N$, set $N_i^\alpha(S) = \{\mathbf{i} \in N^\alpha \mid \operatorname{car}(\mathbf{i}, i) = S \cup \{i\}\}$. The theorem then "claims" that $|N_i^\alpha(S)| = f(s, \alpha)$. We prove the claim by induction on s. Fix α.

Induction basis: For $s = 0$, i.e., $S = \emptyset$, we have $|N_i^\alpha(S)| = |\{(i, i, \ldots, i)\}| = 1 = f(0, \alpha)$.

Induction hypothesis (IH): Let the claim hold for all $s \leq \mathfrak{s} < \alpha$.

Induction step: Fix $S \subseteq N \setminus \{i\}$, $|S| = \mathfrak{s} + 1$. The number of $\mathbf{i} \in N^\alpha$ such that $\operatorname{car}(\mathbf{i}, i) \subseteq S \cup \{i\}$ is $(\mathfrak{s} + 1 + 1)^\alpha$. To obtain $|N_i^\alpha(S)|$, we have to subtract the numbers $|N_i^\alpha(T)|$, $T \subsetneq S$. By the induction hypothesis, we have

$$
\begin{aligned}
|N_i^\alpha(S)| &\overset{IH}{=} (\mathfrak{s}+1+1)^\alpha - \sum_{t=0}^{\mathfrak{s}} \binom{\mathfrak{s}+1}{t} \cdot f(t, \alpha) \\
&\overset{(G.2)}{=} (\mathfrak{s}+1+1)^\alpha - \sum_{k=0}^{\mathfrak{s}} \binom{\mathfrak{s}+1}{k} \cdot \sum_{t=0}^{k} \left[(-1)^{k-t} \cdot \binom{k}{t} \cdot (t+1)^\alpha\right] \\
&= (\mathfrak{s}+1+1)^\alpha - \sum_{t=0}^{\mathfrak{s}} (t+1)^\alpha \sum_{k=t}^{\mathfrak{s}} \left[(-1)^{k-t} \cdot \binom{\mathfrak{s}+1}{k} \cdot \binom{k}{t}\right] \\
&= (\mathfrak{s}+1+1)^\alpha - \sum_{t=0}^{\mathfrak{s}} \binom{\mathfrak{s}+1}{t} \cdot (t+1)^\alpha \sum_{k=t}^{\mathfrak{s}} \left[(-1)^{k-t} \cdot \binom{\mathfrak{s}+1-t}{k-t}\right] \\
&= (\mathfrak{s}+1+1)^\alpha - \sum_{t=0}^{\mathfrak{s}} \binom{\mathfrak{s}+1}{t} \cdot (t+1)^\alpha \sum_{k=0}^{\mathfrak{s}-t} \left[(-1)^{k} \cdot \binom{\mathfrak{s}+1-t}{k}\right] \\
&= (\mathfrak{s}+1+1)^\alpha - \sum_{t=0}^{\mathfrak{s}} \binom{\mathfrak{s}+1}{t} \cdot (t+1)^\alpha \cdot \left(-(-1)^{\mathfrak{s}+1-t}\right) \\
&= (\mathfrak{s}+1+1)^\alpha + \sum_{t=0}^{\mathfrak{s}} \binom{\mathfrak{s}+1}{t} \cdot (t+1)^\alpha \cdot (-1)^{\mathfrak{s}+1-t} \\
&= \sum_{t=0}^{\mathfrak{s}+1} (-1)^{\mathfrak{s}+1-t} \cdot \binom{\mathfrak{s}+1}{t} \cdot (t+1)^\alpha,
\end{aligned}
$$

which concludes the proof. □

The theorem now follows from Lemma 17 by the following fact. For all $\alpha \in \mathbb{N}$, $N \in \mathcal{N}$, $i \in N$, and $S \subseteq N \setminus \{i\}$ such that $s \leq \alpha$, the set $\{\mathbf{i} \in (N \setminus \{i\})^s \mid \operatorname{car}(\mathbf{i}) = S\}$ has a cardinality of $s!$. Finally, $f(s, \alpha) = |N_i^\alpha(S)| > 0$ implies $g(s, \alpha) > 0$.

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