Coalitional Manipulations and Immunity of the Shapley Value

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November 1, 2023

Abstract

We consider manipulations in the context of coalitional games, where a coalition aims to increase the total payoff of its members. An allocation rule is immune to coalitional manipulation if no coalition can benefit from internal reallocation of worth among its subcoalitions (reallocation-proofness), and if no coalition can benefit from a lower worth (weak coalitional monotonicity). Replacing additivity in Shapley's original characterization by these requirements yields a new foundation of the Shapley value, i.e., it is the unique efficient and symmetric allocation rule that awards nothing to a null player and is immune to coalitional manipulations. We further find that for efficient allocation rules, reallocation-proofness is equivalent to constrained marginality, a weaker variant of Young's marginality axiom. Our second characterization improves upon Young's characterization by weakening the independence requirement intrinsic to marginality.

Keywords: Shapley value; reallocation-proofness; constrained monotonicity; non-manipulability; coalitional manipulation; machine learning; advertisement; cost allocation; game theory

JEL Classification: C71, D24, D70.

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1 Introduction

In recent years, cooperative game theory and its solution methods have expanded beyond traditional realms like cost-sharing (Shubik, 1962; Littlechild and Owen, 1973; Tijs and Driessen, 1986; Gopalakrishnan et al., 2021) and property rights remuneration (Hart and Moore, 1990; Tauman and Watanabe, 2007). Their application in statistics for identifying important variables (Lipovetsky and Conklin; Shorrocks, 2012), in machine learning for interpreting prediction models (Lundberg and Lee, 2017; Lundberg et al., 2020), and in marketing for attributing online advertisers' impact on customer conversion (Dalessandro et al.; Berman; Singal et al., forthcoming) demonstrate that today, allocation rules for cooperative games are routinely computed and implemented (Grömping, 2015; Google; GitHub). The most prominent allocation rule for these applications is the Shapley value, which rewards (punishes) a player for a higher (lower) marginal contributions. In fact, this strong monotonicity property is characteristic of the Shapley value (Young, 1985), which is commonly cited as a compelling reason for the widespread adoption of the Shapley value (Shorrocks, 2012; Huettner and Sunder, 2012; Lundberg and Lee, 2017). While strong monotonicity ensures that *individuals* have an incentive to work towards a common goal, less is known about the incentives of *groups of players*.

In this paper, we study *coalitional manipulations*, i.e., modifications of a game with the intention to increase the total payoff accruing to the members of a coalition even as the coalition does not create any additional surplus. We introduce axioms ensuring that allocation rules are immune to such manipulations.

A most basic requirement is *weak coalitional monotonicity* (Zhou, 1991), which ensures that no coalition shall benefit from having a lower worth,¹ while all else remains the same – for example by underreporting their joint contribution to the worth of the grand coalition or by a binding pre-game agreement that reduced their worth understood as the coalition's outside option.

Inspired by Moulin (1987), Moulin (1987), Thomson, Ju et al. (2007), and Ju (2013), we further define *reallocation-proofness* as the requirement that no coalition shall benefit from an internal reallocation of surplus. Here, internal reallocation

¹Following established nomenclature, we refer to the number v(S) that is assigned to coalition S in a coalitional game as the "worth" of this coalition. We use the term "value" when referring to a particular, well-known allocation rule, such as the Shapley value or the equal division value.

means that a coalition manipulates the original game without changing (i) the worth of the manipulating coalition itself, nor (ii) the worth of coalitions of outside players, nor (iii) the synergies between players from the manipulating coalition with outside players. For example a manipulating coalition may be able to misrepresent how its subcoalitions contribute towards the worth of the grand coalition or use binding pregame agreements on how to split up the coalitions' worth among subcoalitions should the grand coalition fail to form – under reallocation proofness, such schemes should not improve the aggregate payoffs to the members of the manipulating coalition.

In the context of statistics, it is not plausible that such a manipulative initiative is the result of features or other model entities "becoming active". Nonetheless, immunity to manipulation is still a plausible requirement in the sense that it prevents a modeler or statistician to inflate the importance of a set of features or a set of model components through a manipulation of the game. This is particular important if the model is otherwise noninterpretable or treated as a blackbox, and its understanding mainly relies on allocation rules from cooperative game theory like the Shapley value.

Consider for instance an XGBoost trained model that predicts a person's health condition based on interventions features (e.g., medicine vs. placebo, special diet plan vs. unregulated diet) and on standard individual characteristics features (e.g., an individual's weight). Unlike a regression model, XGBoost does not give coefficients that show how much a feature's value affects the dependent variable. Instead, this can be done by the Shapley value, which breaks down the contribution of each feature to a person's predicted health condition (Lundberg et al., 2020). A prediction model might appear more desirable to the modeler, say a pharmaceutical company, the higher the sum of Shapley values of interventions features. Reallocationproofness prevents that the modeler could inflate the importance of interventions features through a manipulation, say through different feature engineering, that merely shift explanatory worth from one interventions features to another.

Our main result states that the Shapley value is the unique allocation rule that satisfies

- reallocation-proofness,
- weak coalitional monotonicity,
- efficiency (the sum of payoffs equals the worth of the grand coalition),

- null player (a player's payoff is zero if this player's presence does not affect the worth of any coalition), and
- symmetry (interchangeable players get the same payoff).

The result holds true both on the domain of superadditive games, and on the unrestricted domain of all (possibly not superadditive) cooperative games. The latter might be appropriate, e.g., for applications in statistics.

Notably, the nucleolus Schmeidler (1969) satisfies all these properties except reallocation-proofness. Hence, while the difference between the Shapley value and the nucleolus is often pinned down to the fact that the Shapley value is monotonic while the nucleolus is in the stable, our result offers a new perspective. We argue that the Shapley value guards against coalitions manipulating *within* a game, whereas the nucleolus (and other core selectors) guards against coalitions deviating by *braking away*.

If we restrict attention to efficient allocation rules, then Young's strong monotonicity ensures immunity to manipulation. Specifically, strong monotonicity implies weak coalitional monotonicity. Moreover, an efficient allocation rule satisfies reallocation-proofness if and only if it satisfies *constrained marginality*. The latter requires that a player's payoff remains the same if both this player's marginal contributions *and* the worth of the grand coalition remain the same. Since constrained marginality cannot be applied if the worth of the grand coalition changes, it is a weaker assumption than Young's marginality axiom (an undirected and weaker variant of strong monotonicity). For example, note that the equal division value satisfies constrained marginality but not marginality (nor strong monotonicity).

We obtain another characterization of the Shapley value by means of constrained marginality, weak coalitional monotonicity, efficiency, null player, and symmetry. Our second characterization of the Shapley value sheds light on Young's characterization by replacing strong monotonicity by the requirements of weak coalitional monotonicity, null player (an immediate implication of strong monotonicity given efficiency and symmetry), and constrained marginality.

According to constrained marginality, a player's payoff derives from this player's marginal contributions, but may also depend on the surplus created by other players. This notably weakens the independence requirement intrinsic to Young's marginality.

To see this more lucidly, note that constrained marginality allows a player's payoff to depend on other players' marginal contributions in arbitrary ways, provided that changes in other players' marginal contributions also affect the worth of the grand coalition. Only if a player's marginal contributions as well as the worth created by the other players remain the same, i.e., only if a player's marginal contributions remain the same in *absolute* terms and *relative* to the productivity of the other players, only then does constrained marginality mandate that the player's payoff remains the same.

The paper is organized as follows. In Section 2, we illustrate our axioms with a basic example, and provide formal definitions. In Section 3, we present the main result. In Section 4, we draw a comparison with Young's characterization. Section 5 concludes. The appendix contains all proofs and counterexamples, demonstrating the axioms' independence.

2 Problem formulation

We first illustrate weak coalitional monotonicity and reallocation-proofness by a simple attribution problem in the context of advertising. Then, we provide basic definitions and a establish a formal definition of our new axioms.

2.1 Example with Three Players

A company uses three advertisement services to convert customers: search ads (s), display ads (d), and email ads (e). The (joint) conversion scores for the respective services are given in Figure 1a. In the language of cooperative games, these correspond to the worth of the various coalitions. In this example, email by itself can be perceived as spam and has a negative impact on customer conversion, resulting in a score of -10. However, it works well in conjunction with the other services, yielding scores of 40 and 1 in conjunction with search and display ads, respectively. Combining all three services amounts to a conversion score of 54. For simplicity, we assume no synergies between display and search ads.

Consider the coalition of email and search (or rather their responsible managers) and imagine that they seek to misrepresent contributions, thus modifying the coali-

	25s 20	25 (\$) 20
25 (a) (b) (b) (b) (b) (c)	5 (d) $54 / 4013 (e)$	5 (d) 54 40
1 e	(b) $\{s, e\}$ manip- ulates by internal	$\begin{array}{c} 1 (e) \\ 2 \\ (c) \{s \ e\} \text{manipu-} \end{array}$
(a) Original game;	reattribution, keep-	lates by a reattribu-
signs the following	ing synergies with the outsider d con-	tion that affects syn- ergies with the out-
payoffs: Sh _d = 9	stant: Sh _d = 9	sider d : Sh _d = 7
Sh_{a} = 36	Sh_a = 30	Sh_{s} = 34
$Sh_e = 9$	$\mathrm{Sh}_e~=~15$	$\mathrm{Sh}_e~=~13$

Figure 1: Manipulations by coalition $\{s, e\}$; changes compared to original values in (a) are bold; the Shapley value is immune to manipulation by internal reattribution that do not affect synergies with outsiders, i.e., it satisfies reallocation-proofness, which ensures that s and e together get no higher payoffs in (b) than in (a); the Shapley value is susceptible to manipulation by reallocation that affects synergies with outsiders, i.e., it fails strong reallocation-proofness, which would ensure that s and e together get no higher payoffs in (c) than in (a).

tional game, in order to increase their total payoff. Suppose that the joint performance of email and search includes the positive effect on conversion of searchtriggered follow-up emails to the amount of, say, 12. If this effect is instead ascribed to email directly, then this yields the game of Figure 1b. Note that for such a manipulation to be interpreted as a reallocation among coalition members, it should have no impact on the total worth of the manipulating coalition, i.e., on the joint conversion score of email and search; it does however increase the joint conversion score of email and display, as the synergies between these services remain unchanged while additional conversions, now ascribed to email alone, are included in conversions ascribed to display and/or email ads. Reallocation-proofness requires that such an internal reallocation within the coalition of email and search shall not increase the total payoff that accrues to email and search.

Next, consider the case when the manipulation by the coalition email and search affects the synergy between email and display adds as in Figure 1c. In that case,

a coalitional manipulation affects the marginal contributions of the outside players (here, the marginal contribution of d to e decreases by 12). Reallocation-proofness does not apply in this case. A stronger assumption, called *strong reallocation-proofness*, precludes an increase in a total payoff of a manipulating coalition even when the coalition is able to affect the marginal contributions of outsiders. The Shapley value does not satisfy strong reallocation-proofness (in the example, the aggregate Shapley values of email and search increase by 2). In fact, turns out that the equal division value is the only efficient and symmetric allocation rule that satisfies strong reallocation-proofness.

Finally, we deal with the situation in which the manipulating coalition is ceteris paribus able to lower its conversion score, say by underreporting its performance, as in Figure 2b. Weak coalitional monotonicity requires that this should not be advantageous to the manipulating coalition, i.e., the total payoff to email and search shall not increase from (a) to (b).



Figure 2: Manipulation by coalition $\{s, e\}$; changes compared to original values in (a) are bold; the Shapley value is immune to manipulation by underreporting, i.e., it satisfies weak coalitional monotonicity, which ensures that s and e together get no higher payoffs in (b) than in (a).

2.2 Basic definitions

We consider coalitional games $v: 2^N \to \mathbb{R}, v(\emptyset) = 0$, where N denotes the player set and v(S) the worth of coalition $S \subseteq N$. The space of all coalitional games on N is denoted by $\overline{\mathbb{V}}$. A game v is superadditive if $v(S \cup T) \ge v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$. The collection of all superadditive coalitional games is denoted \mathbb{V}^s . Moving forward, we simply refer to the domain of games by \mathbb{V} , where the results in this paper are correct if we limit all definitions and the validity of the axioms either to the superadditive domain (read $\mathbb{V} = \mathbb{V}^s$), or if we consider the unrestricted domain (read $\mathbb{V} = \overline{\mathbb{V}}$).

An allocation rule φ maps every coalitional game into payoffs, i.e., φ determines for all $v \in \mathbb{V}$, and $i \in N$ a payoff $\varphi_i(v) \in \mathbb{R}$. The *equal division value* van den Brink (2007) assigns to each player the same share of the grand coalition's worth,

$$\mathrm{ED}_{i}(v) = \frac{v(N)}{|N|}.$$
(1)

The *Shapley value* Shapley (1953) assigns to each player its average marginal contribution,

$$Sh_{i}(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(|N| - 1 - |S|)! |S|!}{|N|!} (v(S \cup \{i\}) - v(S)).$$
(2)

The Shapley value satisfies the following standard axioms – and is characterized by them.

Additivity, A. For all $v, w \in \mathbb{V}$, we have $\varphi(v + w) = \varphi(v) + \varphi(w)$.² Efficiency, E. For all $v \in \mathbb{V}$, we have $\sum_{i \in N} \varphi_i(v) = v(N)$. Null Player, N. For all $v \in \mathbb{V}$ and all $i \in N$ we have

$$v(S \cup \{i\}) = v(S)$$
 for all $S \subseteq N \setminus \{i\}$ \Rightarrow $\varphi_i(v) = 0.$

Symmetry, S. For all $v \in \mathbb{V}$ and all $i, j \in N$ we have

$$v(S \cup \{i\}) = v(S \cup \{j\})$$
 for all $S \subseteq N \setminus \{i, j\}$ \Rightarrow $\varphi_i(v) = \varphi_j(v)$.

Theorem 1 (Shapley (1953)) An allocation rule φ satisfies additivity (**A**), efficiency (**E**), null player (**N**), and symmetry (**S**) if and only if φ is the Shapley value.

While additivity is a desirable technical property, its normative appeal is arguably weaker than that of the other axioms. We next introduce axioms, based on which we will provide a characterization of the Shapley value that does not rely on additivity.

²Here $(v+w) \in \mathbb{V}$ is defined by (v+w)(S) = v(S) + w(S) for all $S \subseteq N$.

2.3 Axioms guaranteeing immunity to manipulation

We consider manipulations by coalitions that can be interpreted as reallocations among its members. Hence, (i) the worth of coalitions that consist only of outside players should be unchanged by the manipulation. Moreover, a reallocations among its members should chancel out for the manipulating coalition as a whole, i.e., (ii) the worth of the manipulating coalition itself, as well as the worth of every larger coalition, should also remain unchanged. The strongest conceivable notion of reallocation-proofness would thus demand that the total payoffs accruing to members of a manipulating coalition do not increase between any two games as long as (i) and (ii) are satisfied.

Strong Reallocation-proofness, R⁺. An allocation rule φ satisfies strong reallocationproofness if for all $v, w \in \mathbb{V}$ and all $M \subseteq N$, we have

$$\begin{bmatrix} v(T) = w(T) \\ \text{for all } T \subseteq N \setminus M \\ \text{and for all } T \supseteq M \end{bmatrix} \Rightarrow \sum_{i \in M} \varphi_i(v) \geqslant \sum_{i \in M} \varphi_i(w).$$

This strong notion of reallocation-proofness turns out to be rather restrictive. In particular, we find that the equal division value is the only efficient and symmetric allocation rule that satisfies strong reallocation-proofness.

Proposition 1 Let $|N| \neq 2$. An allocation rule φ satisfies symmetry (**S**), efficiency (**E**), and strong reallocation-proofness (**R**⁺) if and only if φ is the equal division value.

Proposition 1, thus begs the question how to adjust strong reallocation-proofness in a way that preserves its intuitive interpretation of preventing profitable reallocations. A natural variation is to further reduce the set of feasible manipulations. Since a manipulation seems more plausible if it remains unnoticed by the outside players, we shall limit a coalition's manipulation to *internal* reallocation, i.e., reallocations that have no impact on outside players' contributions. Conversely, where manipulations change contributions of outside players, reallocation-proofness is mute.

Reallocation-proofness, **R**. An allocation rule φ satisfies reallocation-proofness

if for all $v, w \in \mathbb{V}$ and all $M \subseteq N$, we have

$$\begin{bmatrix} v(M) = w(M) & \text{and} \\ v(S \cup T) - v(S) = w(S \cup T) - w(S) \\ \text{for all } S \subseteq M \text{ and for all } T \subseteq N \setminus M \end{bmatrix} \Rightarrow \sum_{i \in M} \varphi_i(v) \ge \sum_{i \in M} \varphi_i(w).$$

Reallocation-proofness requires that the total payoff to a coalition does not increase whenever this coalition reattributes worth internally, i.e., such that (i) the worth of the manipulating coalition remains the same; (ii) the worth of every coalition of outside players remains the same $(S = \emptyset)$; and (iii) the surplus created by outside players remains the same.

If we limit the analysis to the domain of superadditive games, $\mathbb{V} = \mathbb{V}^s$, then both the original game v and the manipulated game w have to be superadditive. This additional domain restriction means that reallocation-proofness only applies if the worth attributed by M to its subcoalitions is not too large, i.e., $w(M') + w(M \setminus M') \leq$ v(M) for all $M' \subseteq M$. In particular, a manipulating coalition cannot arbitrarily inflate the worth of subcoalitions or individual members.

Note that reallocation-proofness is implied by efficiency for |N| = 2. Moreover, moving from strong reallocation-proofness to a weaker version enlarges the set of reasonable allocation rules beyond the equal division value. For example, it now includes the Shapley value.³

Lemma 1 The Shapley value satisfies reallocation-proofness (\mathbf{R}) .

Finally, we want to rule out that a coalition can benefit from a lower worth while the game is otherwise unchanged. Formally, this is guaranteed by the following property, introduced by Zhou (1991).

Weak Coalitional Monotonicity, W. An allocation rule φ satisfies weak coalitional monotonicity if for all $v, w \in \mathbb{V}$ and $M \subseteq N$, we have:

$$[v(M) \ge w(M) \quad \text{and} \quad v(S) = w(S) \text{ for all } S \neq M] \qquad \Rightarrow \qquad \sum_{i \in M} \varphi_i(v) \ge \sum_{i \in M} \varphi_i(w)$$

Weak coalitional monotonicity requires that the total payoff to a coalition shall not

³For further examples of rules satisfying reallocation-proofness, see A.4.

increase whenever the worth of this coalition weakly decreases, while the worth of all other coalitions remains the same. It is implied by coalitional monotonicity—a property considered essential by Shubik (1962) to prevent "corporate idiocy" in the context of cost allocation.⁴

Coalitional Monotonicity, CM. An allocation rule φ satisfies coalitional monotonicity if for all $v, w \in \mathbb{V}$ and $M \subseteq N$, we have:

 $[v(M) \ge w(M)$ and v(S) = w(S) for all $S \ne M$ $\Rightarrow \varphi_i(v) \ge \varphi_i(w)$.

Note, however, that weak coalitional monotonicity is a considerably weaker assumption since its implication requires an increase only for the total payoff of the manipulating coalition, not for each individual player. Hence, it is not only at least as plausible and desirable, but it is further satisfied by virtually all allocation rules considered in the literature. In particular, weak coalitional monotonicity is compatible with core selectors such as the nucleolus (Schmeidler, 1969), which fails other monotonicity principles such as coalitional monotonicity or monotonicity in the aggregate (Megiddo, 1974). Indeed, no core selector satisfies coalitional monotonicity (Young, 1985).⁵

3 Immunity to manipulation is characteristic of the Shapley value

Our main result states that preventing coalitional manipulation along the lines of the axioms motivated in the previous section leaves us with a unique allocation rule: the Shapley value.

Theorem 2 Let $|N| \neq 2$. The Shapley value is the unique allocation rule satisfying symmetry (S), null player (N), efficiency (E), weak coalitional monotonicity (W), and reallocation-proofness (R).

⁴Note that the Shapley value satisfies coalitional monotonicity, so that weak coalitional monotonicity can be replaced by coalitional monotonicity in our characterizations Theorems 2 and 4.

⁵The core itself satisfies a set-valued analogue of weak coalitional monotonicity: Consider $v, w \in \mathbb{V}$, and $M \subseteq N$ s.t. $v(M) \ge w(M)$, and v(S) = w(S) for all $S \ne M$. If $\operatorname{Core}(v), \operatorname{Core}(w) \ne \emptyset$, then for every $x \in \operatorname{Core}(w)$ we have either $x \in \operatorname{Core}(v)$ or $\sum_{i \in M} y_i > \sum_{i \in M} x_i$ for all $y \in \operatorname{Core}(v)$.

Reallocation-proofness and weak coalitional monotonicity both rule out profitable coalitional manipulations. Theorem 2 therefore provides a justification for the Shapley value based on strategic considerations. This complements other perspectives in cooperative game theory, in particular the question of whether an allocation rule is stable, i.e., whether it lies in the Core (Gillies, 1953; Monderer et al., 1992). While Core stability rules out profitable coalitional deviations where a coalition might break away, immunity to coalitional manipulation rules out that a coalition can remain with that grand coalition but profitable alter or misrepresent the game being played.

The proof of Theorem 2 is provided together with the proof of Theorem 4 in the appendix. Notably, it is possible to prove the result both for the space of all coalitional games as well as for the restricted domain of superadditive games. The latter is shown in A.3. Finally, the properties in Theorem 2 are independent – counterexamples are given in A.4, where we also shows that there are other allocation rules satisfying all properties for |N| = 2.

4 Comparison to Young's characterization of the Shapley value

A prominent characterization of the Shapley value due to Young (1985) rests on strong monotonicity.

Strong Monotonicity, M⁺. An allocation rule φ satisfies strong monotonicity if for all $v, w \in \mathbb{V}$ and all $i \in N$, we have

$$v(T \cup \{i\}) - v(T) \ge w(T \cup \{i\}) - w(T) \text{ for all } T \subseteq N \setminus \{i\} \implies \varphi_i(v) \ge \varphi_i(w)$$

Strong monotonicity requires that a player's payoff shall not increase whenever this player's marginal contributions to all coalitions weakly decrease. In conjunction with efficiency and symmetry, this property is characteristic of the Shapely value. Young emphasized that a weaker, undirected version of strong monotonicity is sufficient to obtain uniqueness.

Marginality, M. An allocation rule φ satisfies marginality if for all $v, w \in \mathbb{V}$ and

all $i \in N$, we have

$$v(T \cup \{i\}) - v(T) = w(T \cup \{i\}) - w(T) \text{ for all } T \subseteq N \setminus \{i\} \implies \varphi_i(v) = \varphi_i(w).$$

Marginality "is a type of independence condition rather than a monotonicity condition" Young (1985), stipulating that a player's payoff shall *only* depend on this player's marginal contributions.

Theorem 3 (Young (1985)) An allocation rule φ satisfies [strong monotonicity (M^+) or marginality (M)], efficiency (E), and symmetry (S) if and only if φ is the Shapley value.

In view of this theorem, the Shapley value can be considered as *the* allocation rule that reflects a player's merit in a game as measured by this player's marginal contributions. Strong monotonicity further deters individual players from strategically underperforming.

In contrast to strong monotonicity and marginality, weak coalitional monotonicity carries no notion of independence, i.e., it does not require the payoff of a player or of a coalition to be invariant with respect to all changes that do not affect their marginal contributions. Hence, Theorems 2 and 3 are both technically different and support different interpretations. Young's characterization emphasizes individual incentives, whereas our characterization emphasizes the non-profitability of coalitional manipulations. Note, however, that strong monotonicity implies coalitional monotonicity (Young, 1985), which in turn implies weak coalitional monotonicity.

Even though the marginality principle has "a long tradition in economic theory" (de Clippel and Serrano, 2008), it might be counterintuitive to require the extend of independence that is embodied in Young's marginality axiom, i.e., that a player's payoff cannot depend on other factors beyond this player's own marginal contribution, and as such will be be disconnected by assumption from the wider economic environment.⁶ In particular, we may be cautious to insist on the marginality axioms in scenarios in which the worth of the grand coalition changes dramatically. This

⁶Specifically, marginality is equivalent to requirement that a player's payoff shall only depend on this player's dividends (see Eq. (4) in the appendix for a formal definition of dividends), i.e., marginality is equivalent to $[d^v(S) = d^w(S)$ for all $S \ni i] \Rightarrow \varphi_i(v) = \varphi_i(w)$. Hence, marginality rules out that the dividends $d^v(T), T \subseteq N \setminus \{i\}$ can have any influence on player *i*'s payoff.

leads us to the following weakening of marginality.

Constrained Monotonicity, CM. An allocation rule φ satisfies constrained marginality if for all $v, w \in \mathbb{V}$ and all $i \in N$, we have

$$\left[\begin{array}{l} v(N) = w(N) \quad \text{and} \\ v(T \cup \{i\}) - v(T) = w(T \cup \{i\}) - w(T) \text{ for all } T \subseteq N \setminus \{i\} \end{array}\right] \Rightarrow \varphi_i(v) = \varphi_i(w).$$

Similar to marginality, constrained marginality requires that a player's payoff depends on this player's marginal contributions; however, it may also depend on the worth of the grand coalition. Consequently, a player's payoff may increase, even if this player's *absolute* marginal contributions remain the same – given that this player's marginal contributions increase *relative* to the productivity of the others. Moreover, constrained marginality is compatible with the equal division value. In this sense, it is a clearly weaker assumption than marginality.

Interestingly, constrained marginality is closely connected to reallocation-proofness. In fact, both principles are equivalent for rules that satisfy efficiency.

Lemma 2 Let φ satisfy efficiency (**E**). Then, φ satisfies constrained marginality (**CM**) if and only if φ satisfies reallocation-proofness (**R**).

As a consequence, if we restrict ourselves to efficient allocation rules, then strong monotonicity implies both weak coalitional monotonicity and reallocation-proofness. In other words, ensuring that no individual has incentives to strategically underperform already paves the way for immunity to coalitional manipulations. Moreover, we obtain an equivalent result to Theorem 2, if we replace reallocation-proofness by constrained marginality.

Theorem 4 Let $|N| \neq 2$. An allocation rule φ satisfies symmetry (**S**), null player (**N**), efficiency (**E**), weak coalitional monotonicity (**W**), and constrained marginality (**CM**) if and only if $\varphi =$ Sh.

Note that all properties in Theorem 4 are independent of each other. This indicates once more that constrained marginality is considerably weaker than marginality. While the latter in conjunction with symmetry and efficiency implies null player, this is not the case for constrained marginality nor for weak coalitional monotonicity. Moreover, since both weak coalitional monotonicity and constrained marginality are implied by strong monotonicity, Theorem 4 sheds light on the many consequences of the latter. Note that the null player property is immediate from Young's axioms, so that Theorem 3 is immediate from Theorem 4 for $|N| \neq 2$. Figure 3 summarizes the mentioned Theorems and the relationship of the utilized axioms.



Figure 3: Four characterizations of the Shapely value; arrows indicate implications of the axioms (see Lemma 2 for equivalence of reallocation-proofness and constrained marginality in presence of efficiency; the other implications follow from Young (1985)).

Finally, we shall compare our result to Casajus and Huettner (2014), who give a characterization of the convex mixtures of the Shapley value and the equal division value (the class of egalitarian Shapley values (Joosten, 1996)) based on weak monotonicity, which was introduced by van den Brink et al. (2013).

Weak Monotonicity, M⁻. An allocation rule φ satisfies weak monotonicity if for all $v, w \in \mathbb{V}$ and all $i \in N$, we have

$$\left[\begin{array}{cc} v(N) \ge w(N) \quad \text{and} \\ v(T \cup \{i\}) - v(T) \ge w(T \cup \{i\}) - w(T) \text{ for all } T \subseteq N \setminus \{i\} \end{array}\right] \quad \Rightarrow \quad \varphi_i(v) \ge \varphi_i(w)$$

Weak monotonicity is a directed variant of constrained marginality. It is a stronger requirement than constrained marginality and, it also implies (weak) coalitional monotonicity. The reverse is not true, which indicates that weak monotonicity – despite its name – is not an innocuous assumption.⁷ Indeed, together with symmetry and efficiency, weak monotonicity already narrows down the set of allocation rules considerably.

Theorem 5 (Casajus and Huettner (2014)) Let $|N| \neq 2$. An allocation rule φ satisfies symmetry (**S**), efficiency (**E**), and weak monotonicity (**M**⁻) if and only if there is an $\alpha \in [0, 1]$ such that $\varphi = \alpha Sh + (1 - \alpha)ED$.

Clearly, adding the null player property to Theorem 5 singles out the Shapley value, because it is the only egalitarian Shapley value that always gives nothing to the null player.

Corollary 1 Let $|N| \neq 2$. An allocation rule φ satisfies symmetry (**S**), efficiency (**E**), weak monotonicity (**M**⁻), and null player (**N**), if and only if $\varphi =$ Sh.

At first glance, Theorem 4 strongly resembles Corollary 1, where we have the "directionality" in weak coalitional monotonicity. However, we remark that we do not know of a proof that directly implies weak monotonicity from the axioms of Theorem 4 other than the entire uniqueness proof of Theorem 4. In this sense, Theorem 4 does not simply follow from Corollary 1. Moreover, it is not clear how the proof of Theorem 5 by Casajus and Huettner (2014) can be adapted to proceed within the important subclass of superadditive games.

5 Concluding Remarks

We study manipulations of cooperative games, where a coalition of players aims to increase the total payoffs accruing to its members. Specifically, we investigate the consequences on the payoffs assigned by the Shapley value or alternative allocation rules. An allocation rule is immune to coalitional manipulation if no coalition can

⁷For example, the allocation rule given by $\varphi = 2\text{Sh} - \text{ED}$ satisfies coalitional monotonicity, weak coalitional monotonicity (**W**), constrained marginality (**CM**), and reallocation-proofness (**RP**), but not weak monotonicity (**M**⁻).

benefit from internal reallocation of surplus (reallocation-proofness), and if no coalition can benefit from underreporting or otherwise reducing its worth while all else remains the same (weak coalitional monotonicity).

Replacing additivity in Shapley's original characterization by reallocation-proofness and weak coalitional monotonicity yields a new characterization of the value. Our characterization results are valid when allocation rules and axioms apply either to the domain of all coalitional games, or to the domain of superadditive games. The latter, restricted, domain not only supports our focus on efficient allocation rules, but also makes reallocation-proofness particularly desirable: as long as it remains within the class of superadditive games when reallocating worth among its subcoalitions, a manipulating coalition can realize the purported subcoalitions' worths by (covertly) staying together and distributing its own worth. As this renders such manipulations unobservable to outsiders, preventing them may only be achieved by ensuring that they are not in the interest of the coalition itself, i.e., by applying a rule that is reallocation-proof.

In this paper, we focus on the interpretation of coalitional manipulation by players. For applications of the Shapley value in statistics and machine learning, where features (or model components) take the role of players, immunity to manipulation is also a sound requirement. It puts limits to the extend in which a modeler or statistician can inflate the importance (measured by the Shapley value or an alternative allocation rule) of a set of features through a manipulation of the game. This aspect becomes especially crucial when the model in question is noninterpretable or treated as a blackbox, and when its assessment relies mainly on the Shapley value. Taking a model agnostic approach on the level of the cooperative game, we demonstrate that there is no general and plausible alternative allocation rule that can guarantee a higher degree of immunity to coalitional manipulation than the Shapley value. Nonetheless, the Shapley value is susceptible to coalitional manipulations if that manipulation affect the synergies of a set of features with the outside features. While this adds clarity to the question of how to manipulate the Shapley value, it also calls for further studies about manipulations of the game in specific applications.

It turns out that Young's (1985) strong monotonicity not only implies weak coalitional monotonicity, but in conjunction with efficiency it also implies reallocationproofness. This allows us to shed more light on Young's characterization based on marginality by weakening this property's notion of independence. Marginality requires a player's payoff to stay the same if this player's marginal contributions stay the same, regardless of the productivity of the other players. We show that reallocation-proofness can be replaced by constrained marginality in our characterization. The latter is a weakening of marginality, in the sense that it only applies if also the value of the grand coalition remains the same, i.e., constrained marginality requires a player's payoff to stay the same if this player's marginal contributions both absolutely and relatively to the other players' productivity stay the same.

Reallocation-proofness and weak coalitional monotonicity ensures that an allocation rule is immune against coalitional manipulations. This complements other strategic considerations in cooperative game theory, in particular the question of whether an allocation is stable or lies in the Core (Gillies, 1953; Monderer et al., 1992). While Core stability ensures stability against coalitional *deviation after* allocating payoffs, immunity to coalitional manipulation ensures stability against coalitional *manipulation before* allocating payoffs.

The concept of reallocation-proofness is also helpful in view of the "Nash program", i.e., the attempt to connect non-cooperative game theory and cooperative game theory, in particular through implementations of the Shapley value or other allocation rules via non-cooperative games (see, e.g., Macho-Stadler et al. (2007), Mc-Quillin and Sugden (2016), Brügemann et al. (2018)). Implementing an allocation rule appears more plausible if it is immune to coalitional manipulation; moreover, we hope that our results help to connect cooperative game theory to the study of mechanisms that rely on some concept of coalitional strategy-proofness.

A Appendix

We first introduce further notation. Thereafter, we provide the proofs, modifications thereof necessary to remain within the superadditive domain, and counterexamples.

A.1 Additional Notation

For ease of notation, we denote |N| = n. If no confusion arises, we omit braces around singletons. The marginal contribution of entity i to coalition $S \subseteq N \setminus i$ is denoted by $\partial_i v(S)$,

$$\partial_i v(S) = v(S \cup i) - v(S). \tag{3}$$

We say that two players $i, j \in N$ are symmetric if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, i.e., if $\partial_i v(S) = \partial_j v(S)$ for all $S \subseteq N \setminus \{i, j\}$. Let $\mathbb{V}^* \subseteq \overline{\mathbb{V}}$ denote the set of symmetric games, i.e., all players are symmetric to each other for $v \in \mathbb{V}^*$. The null game $\mathbf{0} \in \mathbb{V}^*$ is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$.

For a given $v \in \overline{\mathbb{V}}$, the dividends (also known as Möbius inverse Rota) are recursively given by $d^{v}(\emptyset) = 0$, and

$$d^{v}(S) = v(S) - \sum_{R \subsetneq S} d^{v}(R) \quad \text{for all } S \subseteq N.$$
(4)

Let u_T denote the unanimity game given by $u_T(S) = 1$ if $T \subseteq S$ and otherwise $u_T(S) = 0$. It is well-known that every game has the unique representation in unanimity games,

$$v = \sum_{T \subseteq N} d^v(T) u_T.$$
(5)

It is well-known that the Shapley value assigns to each player an equal share of the dividends this player helps to create, i.e.,

$$\operatorname{Sh}_{i}(v) = \sum_{S \subseteq N \text{ s.t. } i \in S} \frac{d^{v}(S)}{|S|}.$$
(6)

The number of non-vanishing terms in Eq. (5) is denoted by #v, and given by

$$\#v = |\{T \subseteq N \mid d^{v}(T) \neq 0\}|.$$
(7)

Denote the set of players who are contained in every coalition with non-vanishing dividend in v by R(v),

$$R(v) = \{i \in N \mid d^{v}(T) \neq 0 \Rightarrow i \in T\}.$$
(8)

Note that R(v) = N implies $d^v(T) = 0$ for $T \neq N$, i.e., $v = \lambda u_N$ for some $\lambda \in \mathbb{R}$. For $x \in \mathbb{R}^n$, $\bar{x} = \sum_{i \in N} x_i/n$ denotes the average. For $x \in \mathbb{R}^n$ and $v \in \bar{\mathbb{V}}$, we define the game $(v+x) \in \overline{\mathbb{V}}$ as the sum of v and the modular game $\sum_{i \in N} x_i u_i$,

$$(v+x)(S) = v(S) + \sum_{i \in S} x_i$$
 for all $S \subseteq N$.

Finally, we write of $\varphi(x)$ instead of $\varphi(\mathbf{0} + x)$.

A.2 Proofs

Whenever (strong) reallocation-proofness applies to a manipulation, then it also applies to the inverse manipulation. Hence, we can imply equality of total payoffs.

A.2.1 Proof of Proposition 1 on $\overline{\mathbb{V}}$

Consider $|N| \ge 3$ and an allocation rule φ that satisfies **E**, **S**, and **R**⁺. Note that **R**⁺ applied to $M = N \setminus \{i\}$ in in combination with **E** give

$$\begin{bmatrix} v(\{i\}) = w(\{i\}) \\ v(N \setminus \{i\}) = w(N \setminus \{i\}) \\ v(N) = w(N) \end{bmatrix} \Rightarrow \varphi_i(v) = \varphi_i(w)$$
(9)

Towards a contradiction, assume that φ is not the equal division value. Then there exists a game $v \in \overline{\mathbb{V}}$ and player $i \in N$ such that $\varphi_i(v) \neq v(N)/n$. Now, consider the game w constructed as follows:

$$w(S) = \begin{cases} v(N), & S = N\\ v(\{i\}) + (v(N \setminus \{i\}) - v(\{i\})) \frac{|S| - 1}{n - 2}, & S \subseteq N \end{cases}$$
(10)

As all players in w are symmetric, **E** and **S** imply $\varphi_i(w) = w(N)/n = v(N)/n$. Yet, applying (9) gives $\varphi_i(v) = \varphi_i(w)$; a contradiction to $\varphi_i(v) \neq v(N)/n$.

A.2.2 Proof of Lemma 1

Note that reallocation-proofness has an equivalent formulation referring to dividends.

Reallocation-proofness, R. For all $v, w \in \mathbb{V}$ and $M \subseteq N$ we have:

$$\begin{bmatrix} \sum_{T \subseteq M} d^v(T) = \sum_{T \subseteq M} d^w(T) \text{ and } \\ d^v(S) = d^v(S) \text{ if } S \cap (N \setminus M) \neq \emptyset \end{bmatrix} \Rightarrow \sum_{i \in M} \varphi_i(v) = \sum_{i \in M} \varphi_i(w).$$

With this, we can easily see that the Shapley value satisfies reallocation-proofness. Using the formula $\operatorname{Sh}_i(v) = \sum_{T \subseteq N \text{ s.t. } i \in T} d^v(T)/|T|$, we get for all $S \subseteq N$,

$$\sum_{i \in S} \operatorname{Sh}_i(v) = \sum_{T \subseteq N \text{ s.t. } T \subseteq S} d^v(T) + \sum_{R \subseteq N \text{ s.t. } R \cap (N \setminus S) \neq \emptyset} \frac{|S \cap R|}{|R|} d^v(R).$$

A.2.3 Proof of Lemma 2

 $\begin{array}{ll} \boldsymbol{E} \mbox{ and } \boldsymbol{R} \mbox{ imply } \mathbf{CM} \colon \mbox{Let } v, w \in \mathbb{V} \mbox{ and } i \in N \mbox{ are such that } \partial_i v(S) = \partial_i w(S) \mbox{ forall} S \subseteq N \setminus i \mbox{ and } v(N) = w(N). \mbox{ Then, we have } v(N \setminus i) = w(N \setminus i), \mbox{ and applying } \mathbf{R} \mbox{ with } M = N \setminus i \mbox{ gives } \sum_{k \in N \setminus i} \varphi_j(v) = \sum_{k \in N \setminus i} \varphi_k(w). \mbox{ With } \mathbf{E}, \mbox{ we then get } \varphi_i(v) = \varphi_i(w). \mbox{ } \mathbf{E} \mbox{ and } \mbox{ CM imply } \mathbf{R} \colon \mbox{ Let } v, w \in \mathbb{V} \mbox{ and } M \subseteq N \mbox{ are such that } \sum_{T \subseteq M} d^v(T) = \end{array}$

E and **CM** imply **R**: Let $v, w \in \mathbb{V}$ and $M \subseteq N$ are such that $\sum_{T \subseteq M} d^v(T) = \sum_{T \subseteq M} d^w(T)$ and $d^v(S) = d^v(S)$ if $S \cap (N \setminus M) \neq \emptyset$. This implies for all $j \in N \setminus M$ and all $S \ni j$ that $d^v(S) = d^v(S)$, and since for all $T \subseteq N \setminus j$ we have

$$\partial_j v(T) \stackrel{(3),(5)}{=} \sum_{T' \subseteq (T \cup \{i\})} d^v(T') u_{T'} - \sum_{T' \subseteq T} d^v(T') u_{T'} = \sum_{S \subseteq T: S \ni i} d^v(S) u_{T'},$$

we get $\partial_j v(T) = \partial_j w(T)$ for all $j \in N \setminus M$ and all $T \subseteq N \setminus j$. Since further

$$v(N) = \sum_{S:S \cap (N \setminus M) \neq \emptyset} d^v(S) + \sum_{T \subseteq M} d^v(T) = \sum_{S:S \cap (N \setminus M) \neq \emptyset} d^w(S) + \sum_{T \subseteq M} d^w(T) = w(N),$$

we can apply **CM** to all players $j \in N \setminus M$ and obtain $\varphi_j(v) = \varphi_k(w)$ for all $j \in N \setminus M$. Finally, with efficiency we get

$$\sum_{i \in M} \varphi_i(v) \stackrel{\mathbf{E}}{=} v(N) - \sum_{j \in N \setminus M} \varphi_j(v) = w(N) - \sum_{j \in N \setminus M} \varphi_j(w) \stackrel{\mathbf{E}}{=} \sum_{i \in M} \varphi_i(w),$$

which completes the proof.

A.2.4 Proof of Theorem 4

It is obvious from the definition of the Shapley value (2) that the above properties are satisfied by the Shapley value. Conversely, let φ satisfy **S**, **N**, **E**, **W**, and **CM**.

Claim 1 For all $v \in \mathbb{V}$, $x, y \in \mathbb{R}^n$ and $i, j, k \in N$ s.t. i, j, and k are symmetric to each other in v: $[\bar{x} = \bar{y} \text{ and } x_i = y_k] \Rightarrow \varphi_i(v + x) = \varphi_k(v + y).$

Let $\bar{x} = \bar{y} = \mu$ and $x_i = y_k = a$. Define $x', y' \in \mathbb{R}^n$ as follows:

$$\begin{aligned} x'_i &= a, \qquad x'_\ell = \frac{n\mu - a}{n - 1} \quad \text{for} \ell \neq i \\ y'_k &= a, \qquad y'_\ell = \frac{n\mu - a}{n - 1} \quad \text{for} \ell \neq k \end{aligned}$$

Thus, for any $j \in N \setminus \{i, k\}$, we find:

$$\varphi_i(v+x) \stackrel{\mathbf{CM}}{=} \varphi_i(v+x') \stackrel{\mathbf{E}}{=} (v+x')(N) - \varphi_j(v+x') - \sum_{\ell \in N \setminus \{i,j\}} \varphi_\ell(v+x')$$

$$\varphi_k(v+y) \stackrel{\mathbf{CM}}{=} \varphi_k(v+y') \stackrel{\mathbf{E}}{=} (v+y')(N) - \varphi_j(v+y') - \sum_{\ell \in N \setminus \{j,k\}} \varphi_\ell(v+y')$$

The right hand side of both equations is the equal because: (v+x')(N) = (v+y')(N); $\varphi_j(v+x') = \varphi_j(v+y')$ by **CM**; $\varphi_j(v+x') = \varphi_\ell(v+x')$ for all $\ell \in N \setminus \{i, j\}$ by **S**; and $\varphi_j(v+y') = \varphi_\ell(v+y')$ for all $\ell \in N \setminus \{j, k\}$ by **S**.

 \diamond Claim 1

By Claim 1, we find that for symmetric $v \in \mathbb{V}^*$, expressions of the form $\varphi_j(v+z)$ depend only on v, z_j , and \bar{z} . This motivates the following notation:

$$\Delta_0^{\varphi}(v,\mu,a) = \varphi_i(v+x) - \varphi_k(v+y)$$
for some $v \in \mathbb{V}^*$, $i,k \in N, x, y \in \mathbb{R}^n$
such that $\bar{x} = \bar{y} = \mu, x_i = a, y_k = 0$

$$(11)$$

Next, we show that $\Delta_0^{\varphi}(v, \mu, a)$ is additive in a and therefore homogeneous in a for rational numbers.

Claim 2 For all $q \in \mathbb{Q}, a, b, \mu \in \mathbb{R}$ and $v \in \mathbb{V}^*$: $\Delta_0^{\varphi}(v, \mu, qa + b) = q\Delta_0^{\varphi}(v, \mu, a) + \Delta_0^{\varphi}(v, \mu, b)$.

Define $x, y \in \mathbb{R}^n$ as follows:

$$\begin{array}{rcl} x_i &=& a \\ y_i &=& a+b \end{array} \begin{array}{rcl} x_k &=& b \\ y_k &=& 0 \end{array} \begin{array}{rcl} x_j &=& \frac{n\mu-a-b}{n-2} \\ y_j &=& \frac{n\mu-a-b}{n-2} \end{array} \begin{array}{rcl} \text{for } j \in N \setminus \{i,k\} \\ \text{for } j \in N \setminus \{i,k\} \end{array}$$

Note that $\bar{x} = \mu = \bar{y}$ and (v + x)(N) = (v + y)(N). Thus, for any $j \in N \setminus \{i, k\}$, we find

$$\begin{array}{lll} \Delta_0^{\varphi}(v,\mu,a+b) & \stackrel{(11)}{=} & \varphi_i(v+y) - \varphi_k(v+y) \\ & \stackrel{\mathbf{E},\,\mathbf{S}}{=} & (v+y)(N) - (n-2)\varphi_j(v+y) - \varphi_k(v+y) - \varphi_k(v+y) \\ & \stackrel{\text{Claim 1 for } j}{=} & (v+y)(N) - (n-2)\varphi_j(v+x) - \varphi_k(v+y) - \varphi_k(v+y) \\ & \stackrel{\mathbf{E},\,\mathbf{S}}{=} & \varphi_i(v+x) - \varphi_k(v+y) + \varphi_k(v+x) - \varphi_k(v+y) \\ & \stackrel{(11)}{=} & \Delta_0^{\varphi}(v,\mu,a) + \Delta_0^{\varphi}(v,\mu,b) \end{array}$$

Finally, it is well-known that any additive function is homogeneous in rational numbers.

 \diamond Claim 2

Next, we argue that Δ_0^{φ} is positive if a is positive and v is the null game.

Claim 3 For all $a, \mu \in \mathbb{R}$: $a \ge 0 \implies \Delta_0^{\varphi}(\mathbf{0}, \mu, a) \ge 0$.

For a = 0, $\Delta_0^{\varphi}(\mathbf{0}, \mu, a) = 0$ follows from definition (11). Now let $a \neq 0$. By **N**, we have $\varphi_k(x) = 0$ if $x_k = 0$. Therefore, (11) simplifies to

$$\Delta_0^{\varphi}(\mathbf{0}, \bar{x}, x_i) = \varphi_i(x) \text{ for all } x \in \mathbb{R}^n.$$
(12)

In particular $\Delta_0^{\varphi}(\mathbf{0}, \mu, \mu) = \varphi_i(\mu, \dots, \mu)$. By **E** and **S**, we further get $\varphi_i(\mu, \dots, \mu) = \mu$, i.e., $\Delta_0^{\varphi}(\mathbf{0}, \mu, \mu) = \mu$.

Now consider the case $\frac{\mu}{a} \in \mathbb{Q}$. By Claim 2, $\frac{\mu}{a}\Delta_0^{\varphi}(\mathbf{0},\mu,a) = \Delta_0^{\varphi}(\mathbf{0},\mu,\mu)$. Hence, $\Delta_0^{\varphi}(\mathbf{0},\mu,a) = \frac{a}{\mu}\mu = a$, i.e.,

$$\Delta_0^{\varphi}(\mathbf{0}, \bar{x}, x_i) = x_i \text{ if } \frac{\bar{x}}{x_i} \in \mathbb{Q}.$$
(13)

Finally, let $x \in \mathbb{R}^n$ be such that $0 < x_i$. Pick some $q \in \mathbb{Q}$ b such that $0 < q(n\bar{x} - x_i) < x_i$, and define $y \in \mathbb{R}^n$ by $y_i = q(n\bar{x} - x_i)$ and $y_k = x_k$ for $k \neq i$. Note that $\frac{\bar{y}}{y_i} = \frac{1+q}{q} \in \mathbb{Q}$. By $\mathbf{W}, \varphi_i(x) > \varphi_i(y)$. Hence, with (12) and (13), we get $\Delta_0^{\varphi}(\mathbf{0}, \bar{x}, x_i) > b > 0$.

 \diamond Claim 3

Next, we show that $\Delta_0^{\varphi}(\mathbf{0}, \mu, a)$ is increasing in a.

 $\textbf{Claim 4} \ \textit{ For all } a', a'', \mu \in \mathbb{R} \text{: } a' \leqslant a'' \Rightarrow \Delta^{\varphi}(\mathbf{0}, \mu, a') \leqslant \Delta^{\varphi}(\mathbf{0}, \mu, a'').$

This follows from $a'' - a' \ge 0$ and

$$\Delta^{\varphi}(\mathbf{0},\mu,a'') \stackrel{\text{Claim 3}}{=} \Delta^{\varphi}(\mathbf{0},\mu,a') + \Delta^{\varphi}(\mathbf{0},\mu,a''-a') \stackrel{\text{Claim 3}}{\geqslant} \Delta^{\varphi}(\mathbf{0},\mu,a').$$

 \diamond Claim 4

Now we can argue that $\Delta_0^{\varphi}(\mathbf{0}, \mu, a)$ is linear in a.

Claim 5 For all $\lambda, a, b, \mu \in \mathbb{R}$: $\Delta_0^{\varphi}(\mathbf{0}, \mu, \lambda a) = \lambda \Delta_0^{\varphi}(\mathbf{0}, \mu, a)$

By Claim 4, $\Delta^{\varphi}(\mathbf{0}, \mu, a)$ is monotonic in a and by Claim 2 we have $\Delta^{\varphi}(\mathbf{0}, \mu, qa) = q\Delta^{\varphi}(\mathbf{0}, \mu, a)$ for rational $q \in \mathbb{Q}$. Since \mathbb{Q} is dense in \mathbb{R} , this proves the claim.

 \diamond Claim 5

Claim 6 For all $x \in \mathbb{R}^n$: $\varphi(x) = \operatorname{Sh}(x)$.

By **N**, we have $\varphi_k(\mathbf{0} + x) = 0$ if $x_k = 0$. Therefore, (11) simplifies to $\Delta_0^{\varphi}(\mathbf{0}, \mu, \mu) = \varphi_i(\mu, \dots, \mu)$. By **E** and **S**, we further get $\varphi_i(\mu, \dots, \mu) = \mu$, i.e., $\Delta_0^{\varphi}(\mathbf{0}, \mu, \mu) = \mu$. By Claim 5, $\Delta_0^{\varphi}(\mathbf{0}, \mu, a) = a$, i.e., $\varphi_i(x) - 0 = x_i$.

 \diamond Claim 6

Next we consider games in which cooperation requires all players.

Claim 7 For all $a, \mu, \lambda \in \mathbb{R}$: $\Delta_0^{\varphi}(\lambda u_N, \mu, a) = \Delta_0^{\varphi}(\mathbf{0}, \mu, a)$.

Let $q \in \mathbb{Q}$, and define $x \in \mathbb{R}^n$ as follows:

$$x_i = qa$$
 $x_k = n\mu - qa$ $x_j = 0$ for $j \in N \setminus \{i, k\}$

We then have

$$q[\Delta_0^{\varphi}(\lambda u_N, \mu, a) - \Delta_0^{\varphi}(\mathbf{0}, \mu, a)]$$

$$\stackrel{\text{Claim 2}}{=} \Delta_0^{\varphi}(\lambda u_N, \mu, qa) - \Delta_0^{\varphi}(\mathbf{0}, \mu, qa)$$

$$\stackrel{(11)}{=} \varphi_i(\lambda u_N + x) - \varphi_j(\lambda u_N + x) - [\varphi_i(x) - \varphi_j(x)]$$

$$= \varphi_i(\lambda u_N + x) - \varphi_i(x) + [\varphi_j(x) - \varphi_j(\lambda u_N + x)].$$

By Claim 1, $\varphi_j(x) - \varphi_j(\lambda u_N + x)$ is independent of the choice of q. Toward a contradiction, suppose $\Delta_0^{\varphi}(\lambda u_N, \mu, a) \neq \Delta_0^{\varphi}(\mathbf{0}, \mu, a)$. Hence, we can find some q such that $\varphi_i(\lambda u_N + x) < \varphi_i(x)$. However, with z given by $z_i = 0$ and $z_k = \lambda/(n-1)$, Claim 6 implies $\varphi_i(x) = \varphi_i(x+z)$. By **CM**, we have $\varphi_i(x+z) = \varphi_i(\lambda u_{N\setminus\{i\}} + x)$. If $\lambda \ge 0$, then **W** further implies $\varphi_i(\lambda u_{N\setminus\{i\}} + x) \le \varphi_i(\lambda u_N + x)$, i.e., $\varphi_i(x) \le \varphi_i(\lambda u_N + x)$ and we arrive at a contradiction. Analogously, for $\lambda \le 0$ we can choose a q such that $\varphi_i(\lambda u_N + x) > \varphi_i(x)$ and construct a contradiction.

 \diamond Claim 7

Claim 8 For all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$: $\varphi(\lambda u_N + x) = \operatorname{Sh}(\lambda u_N + x)$. By Claims 7 and 6, $\Delta_0^{\varphi}(\lambda u_N, \mu, a) = \Delta_0^{\varphi}(\mathbf{0}, \mu, a) = a$. This gives for all $x \in \mathbb{R}^n$

$$x_i - x_j = \Delta_0^{\varphi}(\lambda u_N, \bar{x}, x_i) - \Delta_0^{\varphi}(\lambda u_N, \bar{x}, x_j) = \varphi_i(\lambda u_N + x) - \varphi_j(\lambda u_N + x).$$

Summing up over all $j \in N$ yields

$$nx_{i} - \sum_{j \in N} x_{j} = n\varphi_{i}(\lambda u_{N} + x) - \sum_{j \in N} \varphi_{j}(\lambda u_{N} + x)$$
$$\stackrel{\mathbf{E}}{=} n\varphi_{i}(\lambda u_{N} + x) - \lambda - \sum_{j \in N} x_{j}.$$

Hence, $\varphi_i(\lambda u_N + x) = \lambda/n + x_i = \operatorname{Sh}_i(\lambda u_N + x).$

 \diamond Claim 8

The remainder of the proof establishes $\varphi(v+x) = \operatorname{Sh}(v+x)$ by induction on #v as defined in (7).

Induction basis. For #v = 0, i.e., v = 0, $\varphi(v + x) = Sh(v + x)$ follows from Claim 6.

Induction hypothesis (IH). Suppose $\varphi(v+x) = \operatorname{Sh}(v+x)$ for all $x \in \mathbb{R}^n$ and all $v \in \overline{\mathbb{V}}$ such that $\#v \leq t$.

Induction step. We want to show that

 $\varphi_i(v+x) = \operatorname{Sh}_i(v+x) \text{ for all } x \in \mathbb{R}^n, \text{ all } i \in N, \text{ and all} v \in \overline{\mathbb{V}} \text{ such that} \# v = t+1.$ (14)

Let $(v + x) \in \overline{\mathbb{V}}$ be such that #v = t + 1. In the case of N = R(v), i.e., $v = \lambda u_N$ for some $\lambda \in \mathbb{R}$, Claim 8 establishes (14).

Now consider the case $N \neq R(v)$. Pick any $i \in N \setminus R(v)$ and construct $v^i \in \overline{\mathbb{V}}$ by

$$v^{i} = \sum_{T \subseteq Ns.t.i \in T} d^{v}(T)u_{T}.$$
(15)

Define $y \in \mathbb{R}^n$ as follows:

$$y_i = x_i, \qquad y_k = x_k + \sum_{T \subseteq N \setminus \{i\} \text{ s.t. } k \in T} \frac{d^v(T)}{|T|} = \operatorname{Sh}_k(v - v^i) \quad \text{ for } k \in N \setminus \{i\}.$$

Note that $\#v^i \leq t$, $(v^i + y)(N) = (v + x)(N)$, and $\partial_i(v^i + y)(S) = \partial_i(v + x)(S)$ for all $S \subseteq N \setminus i$. Hence, we have

$$\varphi_i(v+x) \stackrel{\mathbf{CM of }\varphi}{=} \varphi_i(v^i+y) \stackrel{\mathbf{IH}}{=} \operatorname{Sh}_i(v^i+y) \stackrel{(2)}{=} \operatorname{Sh}_i(v+x).$$

Since $i \in N \setminus R(v)$ was chosen arbitrarily, we established (14) for all $i \in N \setminus R(v)$. Now pick some arbitrarily $j \in R(v)$. Choose some $k \in N \setminus R(v)$ and define $z \in \mathbb{R}^n$ as follows:

$$z_k = n\bar{x} - (n-1)x_j \qquad \qquad z_\ell = x_j \qquad \text{for}\ell \in N \setminus k$$

Since any two player in R(v) are symmetric in (v + z), invoking **S**, **E**, and (14) for

 $i \in N \setminus R(v)$, entails

$$|R(v)|\varphi_{j}(v+z) \qquad \stackrel{\mathbf{S} \stackrel{\text{of}}{=} \varphi}{=} \qquad \sum_{\ell \in R(v)} \varphi_{\ell}(v+z)$$
$$\stackrel{\mathbf{E} \stackrel{\text{of}}{=} \varphi}{=} \qquad (v+z)(N) - \sum_{i \in N \setminus R(v)} \varphi_{i}(v+z)$$
$$\stackrel{(14) \text{ for } i \in N \setminus R(v)}{=} \qquad (v+z)(N) - \sum_{i \in N \setminus R(v)} \operatorname{Sh}_{i}(v+z)$$
$$\stackrel{(2)}{=} \qquad |R(v)| \operatorname{Sh}_{j}(v+z).$$

This yields, $\varphi_j(v+z) = \operatorname{Sh}_j(v+z)$. Finally, since (v+z)(N) = (v+x)(N), and $\partial_j(v+z)(S) = \partial_j(v+x)(S)$ for all $S \subseteq N \setminus \{j\}$, **CM** implies $\varphi_j(v+x) = \varphi_j(v+z)$ and $\operatorname{Sh}_j(v+x) = \operatorname{Sh}_j(v+z)$, which establishes $\varphi_j(v+x) = \operatorname{Sh}_j(v+x)$ for $j \in R(v)$.

This completes the proof on the domain $\overline{\mathbb{V}}$. The following section explains which adjustments are necessary for the proof to go through in \mathbb{V}^s .

A.3 Staying within the superadditive domain \mathbb{V}^s

A.3.1 Proof of Proposition 1 on \mathbb{V}^s

To show uniqueness, suppose that φ satisfies \mathbf{R}^+ , \mathbf{E} and \mathbf{S} for games in \mathbb{V}^s . Towards a contradiction, assume that there is $v \in \mathbb{V}^s$ and $i \in N$, s.t. $\varphi_i(v) \neq v(N)/n$. We define three games:

$$\underline{v}^{1}(S) = \begin{cases} \min\{v(T) : |T| = |S|, i \in T\}, & \text{if } i \in S\\ \min\{v(T) : |T| = |S|, i \notin T\}, & \text{if } i \notin S \end{cases};$$
$$\underline{v}^{2}(S) = \begin{cases} \min\{v(T) : |T| = |S|\}, & \text{if } |S| < n - 1\\ \min\{v(T) : |T| = |S|, i \in T\}, & \text{if } |S| = n - 1 \text{ and } i \in S\\ v(S), & \text{if } S = N \setminus \{i\} \text{ or } S = N \end{cases};$$
$$\underline{v}^{3}(S) = \min\{v(T) : |T| = |S|\}.$$

Note that the worth of the grand coalition remains $\underline{v}^1(N) = \underline{v}^2(N) = \underline{v}^3(N) = v(N)$. All players $j \in N \setminus \{i\}$ are symmetric in \underline{v}^1 and in \underline{v}^2 , and all players (including player i) are symmetric in \underline{v}^3 . Moreover, all three games are superadditive.

By (8), $\varphi_i(\underline{v}^1) = \varphi_i(v) \neq v(N)/n$. Thus, by **E** and **S**, $\varphi_j(\underline{v}^1) \neq v(N)/n$ for all $j \in N \setminus \{i\}$. Taking the perspective of any player $j \in N \setminus \{i\}$ and applying (8) to \underline{v}^1 and \underline{v}^2 , we get $\varphi_j(\underline{v}^2) = \varphi_j(\underline{v}^1) \neq v(N)/n$ for all $j \in N \setminus \{i\}$. By **S** and **E**, this implies $\varphi_i(\underline{v}^2) \neq v(N)/n$, i.e., $\varphi_k(\underline{v}^2) \neq v(N)/n$ for all $k \in N$.

Finally let $\underline{k} \in \operatorname{argmin}_{k \in N}\{v(N \setminus \{k\})\}$, so that $\underline{v}^2(N \setminus \{\underline{k}\}) = \underline{v}^3(N \setminus \{\underline{k}\})$. Then, (8) applied to player $\underline{k}, \underline{v}^2$ and \underline{v}^3 , gives $\varphi_{\underline{k}}(\underline{v}^3) = \varphi_{\underline{k}}(\underline{v}^2) \neq v(N)/n$. However, this is a contradiction to $\varphi_k(\underline{v}^3) = v(N)/n$ for all $k \in N$, which follows from **S** and **E** since all players are symmetric in \underline{v}^3 .

A.3.2 Proof of Proposition 4 on \mathbb{V}^s

For all $x \in \mathbb{R}^n$, v + x is superadditive if and only if v is superadditive. The proofs of Claim 1 to Claim 8 stay within the class of superadditive games. For the remaining argument to go through, we add a second induction.

First, define $\mathbb{V}^+ = \{v \in \mathbb{V} \mid d^v(T) \ge 0$ for all $T \subseteq N\} \subseteq \mathbb{V}^s$ as the set of games with non-negative dividends, i.e., a subset of all superadditive games. Note that any game derived from $v^+ \in \mathbb{V}^+$ by deletion of dividends will remain within that set. Hence the induction argument in the proof of Theorem 7 on #v, applied to $v^+ \in \mathbb{V}^+$ instead of of $v \in \mathbb{V}$, i.e., moving to $v_i^+ + y$ by deleting dividends of coalitions not including player *i* and instead distributing them equally among the coalitions members by an adjustment of the modular game, remains within the class of superadditive games. This establishes $\varphi(v + x) = \operatorname{Sh}(v + x)$ for all $v \in \mathbb{V}^+, x \in \mathbb{R}^n$.

Now, take any superadditive game v and define the modular game x^v by $x_i^v = d^v(i)$ for $i \in N$. Further, we define v^* and \tilde{v} by their dividends: $d^{v^*}(T) = d^{\tilde{v}}(T) = 0$ if |T| = 1 while for $|T| \ge 2$ we have $d^{v^*}(T) = \max\{0, \max_{T':|T'|=|T|}d^v(T')\}$ and $d^{\tilde{v}}(T) = d^{v^*}(T) - d^v(T)$. Hence, by (5)

$$v^* = \sum_{T \subseteq N, |T| \ge 2} d^{v^*}(T) u_T$$
 and $\tilde{v} = \sum_{T \subseteq N, |T| \ge 2}^{v} d^{\tilde{v}}(T) u_T$ and $v = v^* + x^v - \tilde{v}$.

Since $d^{v^*}(T) \ge 0$ for all $T \subseteq N$, we have $v^* \in \mathbb{V}^+$ and hence in particular $\varphi(v^* + x^v) = v^*$

 $\operatorname{Sh}(v^* + x^v).$

Moreover, since $d^{\tilde{v}}(T) \geq 0$ for all $T \subseteq N$, deletion of this dividend from \tilde{v} leaves us with a superadditive game, i.e., $v^* + x^v - (\tilde{v} - d^{\tilde{v}}(T)u_T) = v + d^{\tilde{v}}(T)u_T$ is again superadditive. Thus, we can adjust the induction argument for a second loop as follows. The induction index is $\#\tilde{v}$. The induction basis becomes: For $\#\tilde{v} = 0$, i.e., $v = v^* + x^v$, we have $\varphi(v + x) = \operatorname{Sh}(v + x)$. The induction step picks $i \in N \setminus R(\tilde{v})$ and constructs $v^i \in \mathbb{V}$ by

$$v^{i} = v^{*} + x^{v} + \sum_{T \subseteq N \text{s.t.} i \in T} \widetilde{d}^{v}(T) u_{T}.$$

A.4 Counterexamples

The following allocation rule φ^2 satisfies all our axioms **S**, **N**, **E**, **W**, **CM**, and **R** if there are only two players, but differs from the Shapley value: for all $v \in \mathbb{V}(\{i, j\})$, let

$$\varphi_i^2(v) = \begin{cases} \frac{v(\{i,j\})}{2}, & \text{if } v(i) = v(j); \\ \max\{v(i), v(\{i,j\}) - v(j)\}, & \text{if } v(i) > v(j); \\ \min\{v(i), v(\{i,j\}) - v(j)\}, & \text{if } v(i) < v(j). \end{cases}$$

The weighted Shapley value Kalai and Samet (1987) with unequal weights satisfies **N**, **E**, **W**, **CM**, and **R**, but not **S**.

The equal division value, given by $\text{ED}_i(v) = v(N)/n$ for all $v \in \mathbb{V}$ and $i \in N$, satisfies **S**, **E**, **W**, **R**, **CM**, but not **N**.

The null value, given by $\text{Null}_i(v) = 0$ for all $v \in \mathbb{V}$ and $i \in N$, satisfies **N**, **W**, **R**, **CM**, and **S**, but not **E**.

The following allocation rule $\varphi^{\mathbf{W}}$ satisfies **S**, **N**, **E**, **R**, **CM**, but not **W**:

$$\varphi_i^{\mathbf{W}}(v) = \begin{cases} \left(\operatorname{Sh}_i(v) - \frac{d^v(N)}{n} \right) \frac{v(N)}{v(N) - d^v(N)}, & \text{if } d^v(N) > 0 \text{ and } v(N) - d^v(N) > 0 \\ \operatorname{Sh}_i(v), & \text{otherwise} \end{cases}$$

where the dividend $d^{v}(N)$ is defined in (4).

The nucleolus satisfies S, N, E, W, but neither CM nor R.

Acknowledgments

We thank Lars Ehlers for very helpful comments. Support by Deutsche Forschungsgemeinschaft through CRC TRR 190 (project number 280092119) is gratefully acknowledged.

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